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No. XVII.

KEY

TO

SYSTEM OF PRACTICAL MATHEMATICS,

PART I.

CONTAINING

SOLUTIONS OF ALL THE EXERCISES IN THAT WORK.

BY THE

SCOTTISH SCHOOL-BOOK ASSOCIATION.

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PREFACE.

THIS Key is intended partly for the convenience of Teachers, and partly for the use of those who do not enjoy the advantages of an instructor. It has therefore been deemed requisite to give every solution at full length, and in several instances to introduce explanatory *Notes*, and even general *Theorems*, when they flowed naturally from the subject under consideration. With such a guide, united to moderate application, it is hoped that even the solitary student will be able to make his way through the FIRST PART of the SYSTEM of PRACTICAL MATHEMATICS with great benefit and satisfaction.

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KEY.

ALGEBRA.

ART. 20.

1. $8\times3+(4-2)15-\overline{4\times3\times2}=$
$24+30-24=30.$

2. $8(12+15)-2(4\times15-3)=$
$8\times27-2\times57=216-114=102.$

3. $(8-1)(4-1)(3-1)(2+3)=7\times3\times2\times5=210.$

4. $8\sqrt{3\times15+4}-3\sqrt{3\times15+4}=$
$8\sqrt{49}-3\sqrt{49}=8\times7-3\times7=$
$56-21=35.$

5. $64(4\times15-3\times15)-0(2\times8\times2+4)=$
$64(60-45)-0=64\times15=960.$

6. $8^{\frac{1}{3}}(2\times4+15+4)^{\frac{1}{3}}+4^{\frac{1}{2}}(15+3-2)^{\frac{1}{2}}=$
$2\times(27)^{\frac{1}{3}}+2(16)^{\frac{1}{2}}=6+8=14.$

7. Let $x=8$, $y=4$, then $x^3=512$, and $y^3=64$,

$$\therefore \frac{x^3-y^3}{x-y}=\frac{512-64}{8-4}=\frac{448}{4}=112,$$

and $x^2+xy+y^2=64+32+16=112.$

8. Let $a=7$, $x=3$, then $(a+x)=7+3=10$,
and $(a-x)=7-3=4$,
$\therefore (a+x)(a-x)=10\times4=40$,
also $a^2-x^2=49-9=40.$

9. Let $a=4$, $b=3$, and $c=2$, then $a+b-c=4+3-2=5$,
and $(a-b+c)=4-3+2=3$,
$\therefore (a+b-c)(a-b+c)=5\times3=15$,
and $a^2-b^2-c^2+2bc=16-9-4+12=15.$

10. Let $x=4$, $y=3$, then $x^4-y^4=256-81=175$,
and $(x-y)(x^3+x^2y+xy^2+y^3)=64+48+36+27=175.$

In the last four Examples, equal values will always be obtained by taking the same values on each side, whatever *these values may be.*

ADDITION, ART. 21.

1st.	2d.	3d.	4th.
$ax^{\frac{1}{2}}$	$2\ amx^3$	$4\sqrt{x^2+y^2}$	$12\ (x^4-z)^{\frac{1}{3}}$
$\frac{1}{2}ax^{\frac{1}{2}}$	$4\ amx^3$	$7\sqrt{x^2+y^2}$	$(x^4-z)^{\frac{1}{3}}$
$3\ ax^{\frac{1}{2}}$	$7\frac{1}{2}amx^3$	$\sqrt{x^2+y^2}$	$7\ (x^4-z)^{\frac{1}{3}}$
$4\frac{1}{4}ax^{\frac{1}{2}}$	$4\frac{1}{4}amx^3$	$3\sqrt{x^2+y^2}$	$4\ (x^4-z)^{\frac{1}{3}}$
$\frac{3}{4}ax^{\frac{1}{2}}$		$19\sqrt{x^2+y^2}$	$9\frac{1}{4}(x^4-z)^{\frac{1}{3}}$
$9\frac{1}{2}ax^{\frac{1}{2}}$	$17\frac{3}{4}amx^3$	$10\sqrt{x^2+y^2}$	$8\frac{1}{4}(x^4-z)^{\frac{1}{3}}$
		$44\sqrt{x^2+y^2}$	$41\frac{1}{2}(x^4-z)^{\frac{1}{3}}$

ADDITION, ART. 22.

1st.	2d.	3d.	4th.
$a\sqrt{mx}$	$7mpx^2y^3$	$3(x^2-y^2)^{\frac{1}{2}}$	$5(b-c)x^{\frac{1}{3}}$
$-4a\sqrt{mx}$	$-4mpx^2y^3$	$7(x^2-y^2)^{\frac{1}{2}}$	$-7(b-c)x^{\frac{1}{3}}$
$7a\sqrt{mx}$	$-8mpx^2y^3$	$-4(x^2-y^2)^{\frac{1}{2}}$	$+4(b-c)x^{\frac{1}{3}}$
$-3a\sqrt{mx}$	$4mpx^2y^3$	$-8(x^2-y^2)^{\frac{1}{2}}$	$-4(b-c)x^{\frac{1}{3}}$
$+4a\sqrt{mx}$	$-6mpx^2y^3$	$+5(x^2-y^2)^{\frac{1}{2}}$	$7(b-c)x^{\frac{1}{3}}$
$-\frac{a}{3}\sqrt{mx}$	$2mpx^2y^3$	$-2(x^2-y^2)^{\frac{1}{2}}$	$-3(b-c)x^{\frac{1}{3}}$
$4\frac{2}{3}a\sqrt{mx}$	$-5mpx^2y^3$	$(x^2-y^2)^{\frac{1}{2}}$	$2(b-c)x^{\frac{1}{3}}$

ADDITION, ART. 23.

1st.	2d.
$4x^2+8xy+4y^2$	$3ab+4ac-cd$
$3x^2+4xy+y^2$	$4ab+8cd-6bc$
$2x^2-4xy+2y^2$	$-4ab+5cd+6bc$
x^2-y^2	$2ab+4bc$
$10x^2+8xy+6y^2$	$5ab+4ac+12cd+4bc$

3d.	4th.
$4a^2+10ab+7b^2$	$a-3a^{\frac{1}{2}}b^{\frac{1}{2}}+2b$
$3a^2+4ab+b^2$	$3a-4a^{\frac{1}{2}}b^{\frac{1}{2}}-b$
$2a^2+6ab+4b^2$	$-7a+14a^{\frac{1}{2}}b^{\frac{1}{2}}-3b$
$a^2-6ab-7b^2$	$a-4a^{\frac{1}{2}}b^{\frac{1}{2}}+4b$
$10a^2+14ab+5b^2$	$-2a+3a^{\frac{1}{2}}b^{\frac{1}{2}}+2b$

SUBTRACTION, ART. 24.

1st.

$$\begin{array}{c} a^2+2ab+b^2 \\ a^2-2ab+b^2 \\ \hline 4ab \end{array}$$

2d.

$$\begin{array}{cl} x^3+3x^2y+3xy^2+y^3 & \text{minuends.} \\ x^3-3x^2y+3xy^2-y^3 & \text{subtrahends.} \\ \cline{1-1} 6x^2y+2y^3 & \text{remainders.} \end{array}$$

3d.

$$\begin{array}{r} x^3 \qquad\qquad\quad -a^3 \\ x^2-2ax+a^2 \\ \hline x^3-x^2+2ax-a^2-a^3 \end{array}$$

4th.

$$\begin{array}{rl} 3bx^2-4cx+5x^3 & \text{minuends.} \\ -4bx^2+3cx+5x^3 & \text{subtrahends.} \\ \cline{1-1} 7bx^2-7cx \qquad & \text{remainders.} \end{array}$$

MULTIPLICATION, ART. 26.

$$\begin{array}{ccccl} \text{1st.} & \text{2d.} & \text{3d.} & \text{4th.} & \\ 3a^2bc & -\ 5acd & -\ 4a^2c^3 & 7acx & \text{multiplicands.} \\ 7ab^2c^3 & 3bcd & -\ 7bcd & -\ 3acy & \text{multipliers.} \\ \hline 21a^3b^3c^4 & -15abc^2d^2 & 28a^2bc^4d & -21a^2c^2xy & \text{products.} \end{array}$$

ART. 27.

1st.

$$\begin{array}{l} 7a^2+4ab^2+b^2 \\ 3ac \\ \hline 21a^3c+12a^2b^2c+3ab^2c. \end{array}$$

2d.

$$\begin{array}{l} 3a^3+4a^2b-7ac \\ 5a^2b \\ \hline 15a^5b+20a^4b^2-35a^3bc. \end{array}$$

3d.

$$\begin{array}{l} \frac{1}{2}ac+\frac{2}{3}a^2b-\frac{3}{4}ac^2 \\ 4ab^2 \\ \hline 2a^2b^2c+2\frac{2}{3}a^3b^3-3a^2b^2c^2. \end{array}$$

4th.

$$\begin{array}{l} \frac{2}{3}a^2+\frac{4}{3}ab+\frac{2}{9}b^2 \\ 3ab \\ \hline 2a^3b+4a^2b^2+\frac{2}{3}ab^3. \end{array}$$

5th.

$$(7a^2c^2-5bc^2-4ab)\times\tfrac{3}{4}abc=5\tfrac{1}{4}a^3bc^3-3\tfrac{3}{4}ab^2c^3-3a^2b^2c.$$

ART. 28.

1st.

$$\begin{array}{r} a+b \qquad \\ a+b \qquad \\ \hline a^2+ab \qquad \\ ab+b^2 \\ \hline a^2+2ab+b^2. \end{array}$$

2d.

$$\begin{array}{r} a-b \qquad \\ a-b \qquad \\ \hline a^2-ab \qquad \\ -ab+b^2 \\ \hline a^2-2ab+b^2. \end{array}$$

3d.

$$\begin{array}{r} a+b \qquad \\ a-b \qquad \\ \hline a^2+ab \qquad \\ -ab-b^2 \\ \hline a^2 \qquad -b^2. \end{array}$$

4th.

$$\begin{array}{r} a^2+ab+b^2 \qquad \\ a-b \qquad\qquad\qquad \\ \hline a^3+a^2b+ab^2 \qquad \\ -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad\qquad -b^3. \end{array}$$

5*th.*

$$\begin{array}{l}
a^2-ax+x^2\\
a+x\\
\hline
a^3-a^2x+ax^2\\
\quad +a^2x-ax^2+x^3\\
\hline
a^3 \qquad\qquad +x^3.
\end{array}$$

6*th.*

$$\begin{array}{l}
7ab+3ac+4d\\
3a-2c\\
\hline
21a^2b+9a^2c+12ad-14abc-6ac^2-8cd.
\end{array}$$

7*th.*

$$\begin{array}{lllll}
5a & +\ 3x+y & & & \\
3a & +\ 2x & & & \\
\hline
15a^2 & +\ 9ax & +3ay & & \\
 & +10ax & & +6x^2 & +2xy\\
\hline
15a^2 & +19ax & +3ay & +6x^2 & +2xy.
\end{array}$$

8*th.*

$$\begin{array}{l}
x+\frac{1}{2}y-2\\
\frac{1}{4}x+3y\\
\hline
\frac{1}{4}x^2+\frac{1}{8}xy-\frac{1}{2}x\\
\qquad\quad 3xy \qquad +1\frac{1}{2}y^2-6y\\
\hline
\frac{1}{4}x^2+3\frac{1}{8}xy-\frac{1}{2}x+1\frac{1}{2}y^2-6y.
\end{array}$$

9*th.*

$$\begin{array}{l}
x^3+x^2y+xy^2+y^3\\
x-y\\
\hline
x^4+x^3y+x^2y^2+xy^3\\
\quad -x^3y-x^2y^2-xy^3-y^4\\
\hline
x^4 \qquad\qquad\qquad\qquad -y^4.
\end{array}$$

10*th.*

$$\begin{array}{l}
x^4-x^3+x^2-x+1\\
x^2+x-1\\
\hline
x^6-x^5+x^4-x^3+x^2\\
\quad +x^5-x^4+x^3-x^2+x\\
\qquad\quad -x^4+x^3-x^2+x-1\\
\hline
x^6 \qquad -x^4+x^3-x^2+2x-1.
\end{array}$$

11*th.*

$$\begin{array}{l}
ax+bx^2+cx^3\\
\ 1+\ x\ +\ x^2+x^3\\
\hline
ax+bx^2+cx^3\\
\qquad ax^2+bx^3+cx^4\\
\qquad\qquad ax^3+bx^4+cx^5\\
\qquad\qquad\qquad ax^4+bx^5+cx^6\\
\hline
ax+(a+b)x^2+(a+b+c)x^3+(a+b+c)x^4+(b+c)x^5+cx^6.
\end{array}$$

$$\text{or }\begin{array}{r|r|r|r|l}
ax+a & x^2+a & x^3+a & x^4+b & x^5+cx^6\\
+b & +b & +b & +c & \\
 & +c & +c & &
\end{array}$$

12*th.*

Here, in order that the result may be of the same form *as that derived* from actual multiplication, we write, first *the square of the* first quantity; second, twice the produc of the two quantities; and, third, the square of the second quantity; observing always, that if any of the quantities have coefficients, they must be squared and multiplied along with them; hence

$$\begin{aligned}
(x+y)^2 &= x^2+2xy+y^2\\
(2a+x)^2 &= 4a^2+4ax+x^2\\
(a+2b)^2 &= a^2+4ab+4b^2\\
(a^2+x^2)^2 &= a^4+2a^2x^2+x^4\\
(ac+x)^2 &= a^2c^2+2acx+x^2\\
(3x+4y) &= 9x^2+24xy+16y^2.
\end{aligned}$$

13*th*.

The remark on question 12th is applicable to this; the only difference in the results being, that here the product is to be taken with a minus sign.

$$\begin{aligned}
(a-x)^2 &= a^2-2ax+x^2\\
(2a-b)^2 &= 4a^2-4ab+b^2\\
(3x-2y)^2 &= 9x^2-12xy+4y^2\\
(a^2-b^2)^2 &= a^4-2a^2b^2+b^4\\
(ac-y)^2 &= a^2c^2-2acy+y^2\\
(3a^2-4c)^2 &= 9a^4-24a^2c+16c^2.
\end{aligned}$$

14*th*.

On referring to the solution of exercise 3d, in (Art. 28), it will be seen, that when a quantity, increased by another, is multiplied by the same quantity, diminished by that other, the result is the difference of their squares; hence in this exercise the result is always the square of the first quantity, diminished by the square of the second.

$$\begin{aligned}
(a+x)(a-x) &= a^2-x^2\\
(2a+x)(2a-x) &= 4a^2-x^2\\
(3x+2y)(3x-2y) &= 9x^2-4y^2\\
(a^2-b^2)(a^2+b^2) &= a^4-b^4\\
(4a^2-9c^2)(4a^2+9c^2) &= 16a^4-81c^4.
\end{aligned}$$

15*th*.

$$\begin{aligned}
(a-x)(a+x) &= a^2-x^2, \text{ by Theor. 3d.}\\
(a^2-x^2)(a^2+x^2) &= a^4-x^4, \text{ by ditto.}\\
\text{and } (a^4-x^4)(a^4+x^4) &= a^8-x^8, \text{ by ditto.}
\end{aligned}$$

16*th*.

$$\begin{aligned}
(2a+x)(2a-x) &= 4a^2-x^2, \text{ by Theor. 3d.}\\
(4a^2-x^2)(4a^2+x^2) &= 16a^4-x^4, \text{ by ditto.}\\
\text{and } (16a^4-x^4)(16a^4+x^4) &= 256a^8-x^8, \text{ by ditto.}
\end{aligned}$$

17*th.*

$$\begin{array}{l}
x^2+xy+y^2 \\
x-y \\ \hline
x^3+x^2y+xy^2 \\
\quad -x^2y-xy^2-y^3 \\ \hline
x^3 \qquad\qquad -y^3,
\end{array}$$

and $(x^3-y^3)(x^3+y^3)=x^6-y^6$, by Theor. 3d.

18*th.*

$$\begin{array}{l}
x^3-x^2y+xy^2-y^3 \\
x+y \\ \hline
x^4-x^3y+x^2y^2-xy^3 \\
\quad x^3y-x^2y^2+xy^3-y^4 \\ \hline
x^4 \qquad\qquad\qquad -y^4,
\end{array}$$

and $(x^4-y^4)(x^6+y^6)=x^{10}-x^6y^4+x^4y^6-y^{10}$.

Division, Art. 33.

1. $8a^2x^3 \div -2ax^2=-4ax$.
2. $-27p^2mx^4 \div 9pmx=-3px^3$.
3. $-84x^4y^5z^2 \div -12x^3y^2z=7xy^3z$.
4. $120a^2xy^4 \div 30ax^2y^3=\frac{4ay}{x}$.
5. $-40x^{\frac{3}{2}}y^{\frac{1}{2}}z \div 8x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}=-5xz^{\frac{1}{2}}$.
6. $36x^2y^3z \div 20x^{\frac{1}{2}}y^{\frac{3}{2}}z^{\frac{1}{2}}=1\frac{4}{5}x^{\frac{3}{2}}y^{\frac{3}{2}}z^{\frac{1}{2}}$.

Art. 34.

1. $2a^2\underline{)3a^4-6a^3+4a^2-8a}$

   $\frac{3}{2}a^2-3a+2-\frac{4}{a}$.

2. $4ab\underline{)12a^3-36a^2b+36ab^2-12b^3}$

   $\frac{3a^2}{b}-9a+9b-\frac{3b^2}{a}$.

3. $2abx\underline{)48a^4b^2x-36a^2b^2x^2+14ab^3x^4}$

   $24a^3b-18abx+7b^2x^3$.

4. $-10p^2xy\underline{)40p^2x^2y^2-30p^4x^3y^2-60p^2x^3y^4}$

   $-4xy+3p^2x^2y+6x^2y^3$.

5. $4a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}\underline{)7a^3b^{\frac{1}{2}}c^2-12a^{\frac{1}{2}}b^3c-14ab^2c^{\frac{1}{2}}}$

   $\frac{7}{4}a^{\frac{5}{2}}c^{\frac{3}{2}}-3b^{\frac{5}{2}}c^{\frac{1}{2}}-3\frac{1}{2}a^{\frac{1}{2}}b^{\frac{3}{2}}$.

6. $4a^2\underline{)4a^2c^2-8a^2cx+4a^2x^2}$

   $c^2-2cx+x^2$.

ART. 36.

1. $a^2+2ax+x^2)a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5(a^3+3a^2x+3ax^2+x^3$

$$\begin{array}{l}
a^5+2a^4x+a^3x^2 \\
3a^4x+9a^3x^2+10a^2x^3 \\
3a^4x+6a^3x^2+3a^2x^3 \\
3a^3x^2+7a^2x^3+5ax^4 \\
3a^3x^2+6a^2x^3+3ax^4 \\
a^2x^3+2ax^4+x^5 \\
a^2x^3+2ax^4+x^5.
\end{array}$$

2. $a^3-3a^2+3a-1)a^5-5a^4+10a^3-10a^2+5a-1(a^2-2a+1$

$$\begin{array}{l}
a^5-3a^4+3a^3-a^2 \\
-2a^4+7a^3-9a^2+5a \\
-2a^4+6a^3-6a^2+2a \\
a^3-3a^2+3a-1 \\
a^3-3a^2+3a-1
\end{array}$$

3. $x^3+x^2a+xa^2+a^3)x^4 \quad -a^4 \quad (x-a$

$$\begin{array}{l}
x^4+x^3a+x^2a^2+xa^3 \\
-x^3a-x^2a^2-xa^3-a^4 \\
-x^3a-x^2a^2-xa^3-a^4.
\end{array}$$

4. $x^4-x^3y+x^2y^2-xy^3+y^4)x^5 \quad +y^5 \quad (x+y$

$$\begin{array}{l}
x^5-x^4y+x^3y^2-x^2y^3+xy^4 \\
x^4y-x^3y^2+x^2y^3-xy^4+y^5 \\
x^4y-x^3y^2+x^2y^3-xy^4+y^5.
\end{array}$$

5. $x^2-4x-5)9x^6-46x^5+95x^2+150x(9x^4-10x^3+5x^2-30x$

$$\begin{array}{l}
9x^6-36x^5-45x^4 \\
-10x^5+45x^4 \\
-10x^5+40x^4+50x^3 \\
5x^4-50x^3+95x^2 \\
5x^4-20x^3-25x^2 \\
-30x^3+120x^2+150x \\
-30x^3+120x^2+150x.
\end{array}$$

6. $5x^3-4x^2)25x^6-x^4-2x^3-8x^2(5x^3+4x^2+3x+2$

$$\begin{array}{l}
25x^6-20x^5 \\
+20x^5-x^4 \\
20x^5-16x^4 \\
15x^4-2x^3 \\
15x^4-12x^3 \\
10x^3-8x^2 \\
10x^3-8x^2.
\end{array}$$

7. $\tfrac{4}{3}x-2)x^4-\tfrac{15}{6}x^3+x^2+\tfrac{4}{3}x-2(\tfrac{3}{4}x^3-\tfrac{1}{2}x^2+1$

$$\begin{array}{l}
x^4-\tfrac{3}{2}x^3\\
\hline
\quad-\tfrac{4}{6}x^3+x^2\\
\quad-\tfrac{4}{6}x^3+x^2\\
\hline
\qquad\qquad+\tfrac{4}{3}x-2\\
\qquad\qquad\quad\tfrac{4}{3}x-2.\\
\hline
\end{array}$$

The above exercise might also be wrought by multiplying both dividend and divisor by 6 to clear away the fractions, and then dividing the product of the dividend by the product of the divisor.

Art. 38.

The general demonstration of this theorem is as follows:—

Let $x=a+b$, then $x-a=b$,

$x^n=(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2\ldots\ldots+b^n$; from this subtract a^n, and the remainder will be a series of terms, each of which contains $b=(x-a)$, or $x^n-a^n=(a+b)^n-a^n=na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b\ldots\ldots\ldots+b^n$, which is evidently divisible by b, or $(x-a)$, without a remainder.

Art. 39.

In order to demonstrate this theorem generally, put $x+a=s$, then $x=s-a$, and x^n-a^n, will $=s^n-ns^{n-1}a+\frac{n(n-1)}{2}s^{n-2}a^2-\ldots\ldots\pm a^n-a^n$, where the last term of the expanion of $(s-a)$ will be $+$ if n be even, and $-$ if n be odd; $\therefore$ if n be even, the (a^n)'s will destroy each other, but if n be odd, there will remain $2a^n$, which term not being divisible by s or $x+a$, there will be a remainder.

Art. 40.

Let the notation here be the same as in Art. 38, then $x^n+a^n=2a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\ldots.+b^n$. In which the first term $2a^n$ is not divisible by $b=(x-a)$.

Art. 41.

Let the notation here be the same as in Art. 39; then $x^n+a^n=s^n-ns^{n-1}a+\frac{n(n-1)}{2}s^{n-2}a^2-\ldots\ldots\pm a^n+a^n$, which will be divisible by $s=x+a$, if the (a^n)'s have opposite signs, that is, if the last term of the expansion of $(s-a)^n$ be —, which it will be if n be odd. The above expansions are all effected by the binomial theorem.

ART. 43.

1. $3a+\frac{x^2}{a}=\frac{3a\times a+x^2}{a}=\frac{3a^2+x^2}{a}.$

2. $a-b+\frac{ac}{d}=\frac{(a-b)d+ac}{d}=\frac{ad-bd+ac}{d}.$

3. $15a^2+x-\frac{x}{ab}=\frac{(15a^2+x)ab-x}{ab}=\frac{15a^3b+abx-x}{ab}.$

4. $3ax+x^2-\frac{2ax^2-x^3}{bc}=\frac{(3ax+x^2)bc-(2ax^2-x^3)}{bc}$

$=\frac{3abcx+bcx^2-2ax^2+x^3}{bc}.$

5. $x-a-\frac{a^2-ax}{x}=\frac{(x-a)x-(x^2-ax)}{x}=\frac{x^2-ax-x^2+ax}{x}$

$=\frac{x^2-a^2}{x}.$

6. $a+x-\frac{2ax-2x^2}{a-x}=\frac{(a+x)(a-x)-(2ax-2x^2)}{a-x}$

$=\frac{a^2-x^2-2ax+2x^2}{a-x}=\frac{a^2+x^2-2ax}{a-x}.$

ART. 44.

1. $\frac{3ax-x^2}{a}=3x-\frac{x^2}{a}.$

2. $\frac{4x^2+3a}{x}=4x+\frac{3a}{x}.$

3. $\frac{10a^2b+2a^2-3b^2}{2a^2}=5b+1-\frac{3b^2}{2a^2}.$

4. $\frac{12a^4-6a^3+9a^2}{3a^2}=4a^2-2a+3.$

5. $$\begin{array}{l} x^2+2xa+a^2)\,x^3+3x^2a+3xa^2+a^3\,(x+a \\ \qquad\qquad\quad \underline{x^3+2x^2a+xa^2} \\ \qquad\qquad\qquad\quad x^2a+2xa^2+a^3 \\ \qquad\qquad\qquad\quad x^2a+2xa^2+a^3. \end{array}$$

ART. 46.

DEMONSTRATION. The truth of the rule depends upon the two following principles:—

1st, If one quantity measure another, it will also measure any multiple of that quantity. Let x measure y by the units in n, it will measure my by the units in mn.

2d, If a quantity measure two others, it will measure their sum or difference. Let a be contained m times in x, and n times in y; then $ma=x$, and $na=y$; therefore $x\pm y=(m\pm n)a$; that is, a is contained $m\pm n$ times in $x\pm y$, or it measures $x\pm y$ by the units in $m\pm n$.

Now let a and b be the two quantities, and let b be contained in a, p times, with a remainder, c, less than b; again, let c be contained q times in b, with a remainder d, less than c; lastly, let d be contained in c, r times, and no remainder.

$$\begin{array}{l}
b)\,a\,(p\\
\underline{\;\;bp\;\;}\\
\quad c)\,b\,(q\\
\quad\underline{\;\;\;cq\;\;}\\
\qquad d)\,c\,(r\\
\qquad\underline{\;\;\;\;rd\;}\\
\qquad\quad\;\; o
\end{array}$$

Since c contains d r times, and no remainder, d measures any multiple of c (pr 1st), but $b=cq+d=rdq+d=(rq+1)d$, $\therefore d$ measures b; and again, since $a=bp+c=p(rq+1)d+rd=(pqr+p+r)d$; $\therefore d$ measures a, and it has been shown that it measures b; $\therefore d$ is a common measure of a and b.

$$1.\quad\begin{array}{l|l|l|l}
a-2x & a^3-2a^2x+ax^2-2x^3 & a^4-x^4 & a+2x\\
 & a^3\qquad\quad +ax^2 & a^4-2a^3x+a^2x^2-2ax^3 & \\
\hline
 & -2a^2x\qquad\quad -2x^3 & 2a^3x-a^2x^2+2ax^3-x^4 & \\
 & -2a^2x\qquad\quad -2x^3 & 2a^3x-4a^2x^2+2ax^3-4x^4 & \\
\hline
 & & 3x^2)\,3a^2x^2+3x^4 & \\
 & & \qquad a^2+x^2.\qquad \text{Ans.} &
\end{array}$$

$$2.\quad\begin{array}{l|l|l|l}
x^3-y^3 & x^4+x^3y-xy^3-y^4 & x^5+y^5 & x-y\\
 & x^4+x^3y & x^5+x^4y-x^2y^3-xy^4 & \\
\hline
 & \qquad\qquad -xy^3-y^4 & -x^4y+x^2y^3+xy^4+y^5 & \\
 & \qquad\qquad -xy^3-y^4 & -x^4y-x^3y^2+xy^4+y^5 & \\
\hline
 & & x^2y^2)\,x^3y^2+x^2y^3 & \\
 & & \qquad x+y.\qquad \text{Ans.} &
\end{array}$$

$$3.\quad\begin{array}{l|r|r|l}
4x+3 & 48x^2+16x-15 & 24x^3-22x^2+17x-5 & x-1\tfrac{1}{4}\\
 & 48x^2-20x & 2 & \\
\hline
 & 36x-15 & 48x^3-44x^2+34x-10 & \\
 & 36x-15 & 48x^3+16x^2-15x & \\
\hline
 & & -60x^2+49x-10 & \\
 & & -60x^2-20x+18\tfrac{3}{4} & \\
\hline
 & & 69x-28\tfrac{3}{4} & \\
 & & 4 & \\
\hline
 & & 23)\,276x-115 & \\
\hline
 & & \text{Ans. }12x-5 &
\end{array}$$

4. 1st divisor, $x^4+4x^3a+6x^2a^2+4xa^3+a^4$.

The first divisor being multiplied by 20, to make its first term divisible by the first term of the second divisor, becomes the following:—

x(3d quot.)$)20x^4+80x^3a+120x^2a^2+80xa^3+20a^4$

$20x^4+45x^3a+36x^2a^2+11xa^3$

$35x^3a+84x^2a^2+69xa^3+20a^4$

Mult. by 4. $\quad 4$

$7a$(4th quot.)$)140x^3a+336x^2a^2+276xa^3+80a^4$

$140x^3a+315x^2a^2+252xa^3+77a^4$

$3a^2)21x^2a^2+24xa^3+3a^4$

3d divisor, $7x^2+8xa+a^2 \,|\, 7x+a$

$7x^2+7xa$ $\quad$ (7th quot.)

$xa+a^2$

$xa+a^2$

Hence $x+a$ is the greatest common measure sought.

$x^6-c^6 \quad | \quad x^2-4xa+10a^2$

$x^6+4x^5a+6x^4a^2+4x^3a^3+x^2a^4$ $\quad$ (1st quot.)

$-4x^5a-6x^4a^2-4x^3a^3-x^2a^4-a^6$

$-4x^5a-16x^4a^2-24x^3a^3-16x^2a^4-4xa^5$

$+10x^4a^2+20x^3a^3+15x^2a^4+4xa^5-a^6$

$+10x^4a^2+40x^3a^3+60x^2a^4+40xa^5+10a^6$

$-a^3)-20x^3a^3-45x^2a^4-36xa^5-11a^6$, divide by $-a^3$.

$20x^3+45x^2a+36xa^2+11a^3$,(2d divisor.

Mult. by 7. $\quad 7$

$140x^3+315x^2a+252xa^2+77a^3(20x$ (5th quot.)

$140x^3+160x^2a+20xa^2$

$155x^2a+232xa^2+77a^3$

Mult. by 7. $\quad 7$

$1085x^2a+1624xa^2+539a^3(155a$(6th quot.)

$1085x^2a+1240xa^2+155a^3$

$384a)384xa^2+384a^3$

$x+a$ $\quad$ (4th divisor.

ART. 47.

(1.) $\frac{ax+x^2}{3bx-cx}=\frac{(a+x)x}{(3b-c)x}=\frac{a+x}{3b-c}.$

(2.) $\frac{x^2-c^2}{x^2-2cx+c^2}=\frac{(x-c)(x+c)}{(x-c)(x-c)}=\frac{x+c}{x-c}.$

(3.) $\frac{a^2+4ax+4x^2}{a^3-4ax^2}=\frac{(a+2x)(a+2x)}{a(a-2x)(a+2x)}=\frac{a+2x}{a^2-2ax}.$

(4.) $\frac{m^4-n^4}{m^4-2m^2n^2+n^4}=\frac{(m^2-n^2)(m^2+n^2)}{(m^2-n^2)(m^2-n^2)}=\frac{m^2+n^2}{m^2-n^2}.$

(5.) $\frac{4x^2-12ax+9a^2}{8x^3-27a^3}=\frac{(2x-3a)(2x-3a)}{(2x-3a)(4x^2+6ax+9a^2)}$

$=\frac{2x-3a}{4x^2+6ax+9a^2}.$

ART. 48.

(1.) $\frac{3x}{a}\times\frac{3x}{3x}=\frac{9x^2}{3ax}.$

$\frac{2a}{3x}\times\frac{a}{a}=\frac{2a^2}{3ax}.$

$\frac{5c}{x}\times\frac{3a}{3a}=\frac{15ac}{3ax}.$

(2.) $\frac{a-b}{a+b}\times\frac{x(a-b)}{x(a-b)}=\frac{(a-b)^2x}{x(a^2-b^2)}$

$\frac{a+b}{a-b}\times\frac{x(a+b)}{x(a+b)}=\frac{(a+b)^2x}{x(a^2-b^2)}$

$\frac{ab}{x}\times\frac{(a+b)(a-b)}{(a+b)(a-b)}=\frac{ab(a^2-b^2)}{x(a^2-b^2)}.$

(3.) $\frac{5x}{a}\times\frac{(a+b)(a-b)b}{(a+b)(a-b)b}=\frac{5bx(a^2-b^2)}{ab(a^2-b^2)}$

$\frac{3x}{a+b}\times\frac{(a-b)ab}{(a-b)ab}=\frac{3a^2bx-3ab^2x}{ab(a^2-b^2)}$

$\frac{4b}{a-b}\times\frac{ab(a+b)}{ab(a+b)}=\frac{4a^2b^2+4ab^3}{ab(a^2-b^2)}$

$\frac{3x}{b}\times\frac{a(a+b)(a-b)}{a(a+b)(a-b)}=\frac{3a^3x-3ab^2x}{ab(a^2-b^2)}.$

(4) The least common denominator is the least commo multiple of $a+x$, $a-x$, a^2-x^2, which is a^2-x^2; hence

$\frac{7a}{a+x}\times\frac{a-x}{a-x}=\frac{7a^2-7ax}{a^2-x^2}$

$\frac{3x}{a-x}\times\frac{a+x}{a+x}=\frac{3ax+3x^2}{a^2-x^2}$ and $\frac{4ax}{a^2-x^2}.$

(5.) $\frac{a+b}{a-b}\times\frac{a-b}{a-b}=\frac{a^2-b^2}{(a-b)^2}$

$\frac{2a}{a-b}\times\frac{a-b}{a-b}=\frac{2a^2-2ab}{(a-b)^2}$ and $\frac{a+b}{(a-b)^2}$.

(6.) $\frac{a+c}{ac}\times\frac{ac}{ac}=\frac{a^2c+ac^2}{a^2c^2}$

$\frac{a-c}{a^2c}\times\frac{c}{c}=\frac{ac-c^2}{a^2c^2}$

$\frac{a-2c}{ac^2}\times\frac{a}{a}=\frac{a^2-2ac}{a^2c^2}$.

ART. 50.

(1.) By (Art. 49) $\frac{a}{b}=\frac{a^2c}{abc}$, $\frac{b}{c}=\frac{ab^2}{abc}$, $\frac{c}{a}=\frac{bc^2}{abc}$

$\therefore\frac{a^2c}{abc}+\frac{ab^2}{abc}+\frac{bc^2}{abc}=\frac{a^2c+ab^2+bc^2}{abc}$.

(2.) By (Art. 49) $\frac{a+b}{a-b}=\frac{a^2+2ab+b^2}{a^2-b^2}$, $\frac{a-b}{a+b}=\frac{a^2-2ab+b^2}{a^2-b^2}$

$\therefore\frac{a^2+2ab+b^2}{a^2-b^2}+\frac{a^2-2ab+b^2}{a^2-b^2}=\frac{2a^2+2b^2}{a^2-b^2}$.

(3.) The least common multiple of the denominators will be found to be a^3-x^3; the first fraction being multiplied by $\frac{a-x}{a-x}$, becomes $\frac{a^3-ax^2}{a^3-x^3}$; and the second being multiplied by $\frac{a^2+ax+x^2}{a^2+ax+x^2}$, becomes $\frac{a^3+a^2x+ax^2}{a^3-x^3}$; and the third requires no change; hence

$$\frac{a^3-ax^2}{a^3-x^3}+\frac{a^3+a^2x+ax^2}{a^3-x^3}+\frac{a^2+x^2}{a^3-x^3}=\frac{2a^3+a^2x+a^2+x^2}{a^3-x^3}.$$

(4.) $\frac{x}{x-4}\times\frac{x+4}{x+4}=\frac{x^2+4x}{x^2-16}$, and $\frac{x}{x+4}\times\frac{x-4}{x-4}=\frac{x^2-4x}{x^2-16}$

$\therefore\frac{x^2+4x}{x^2-16}+\frac{x^2-4x}{x^2-16}=\frac{2x^2}{x^2-16}$.

(5.) Reduced to the least common denominator, the fractions will be $\frac{6a}{12}$, $\frac{4a}{12}$, $\frac{3a}{12}$, and $\frac{2a}{12}$.

$\therefore\frac{6a}{12}+\frac{4a}{12}+\frac{3a}{12}+\frac{2a}{12}=\frac{15a}{12}=1\frac{1}{4}a$.

ART. 51.

(1.) $\frac{a+x}{2}-\frac{a-x}{2}=\frac{a+x-a+x}{2}=\frac{2x}{2}=x$.

(2.) $\frac{4x}{5}-\frac{3x}{4}=\frac{16x}{20}-\frac{15x}{20}=\frac{x}{20}$.

(3.) $\frac{1}{a-c}\times\frac{a+c}{a+c}=\frac{a+c}{a^2-c^2}$ and $\frac{1}{a+c}\times\frac{a-c}{a-c}=\frac{a-e}{a^2-c^2}$

$\therefore \frac{a+c}{a^2-c^2}-\frac{a-c}{a^2-c^2}=\frac{a+c-a+c}{a^2-c^2}=\frac{2c}{a^2-c^2}.$

(4.) Multiply the first fraction by $\frac{1+x}{1+x}$ and it becomes $\frac{1+x}{1-x^2}$

$\therefore \frac{1+x}{1-x^2}-\frac{1}{1-x^2}=\frac{1+x-1}{1-x^2}=\frac{x}{1-x^2}.$

(5.) $\frac{m+n}{m-n}\times\frac{m+n}{m+n}=\frac{m^2+2mn+n^2}{m^2-n^2}$ and $\frac{m-n}{m+n}\times\frac{m-n}{m-n}$

$=\frac{m^2-2mn+n^2}{m^2-n^2}$; but $\frac{m^2+2mn+n^2}{m^2-n^2}-\frac{m^2-2mn+n^2}{m^2-n^2}$

$=\frac{m^2+2mn+n^2-m^2+2mn-n^2}{m^2-n^2}=\frac{4mn}{m^2-n^2}.$

(6.) $\frac{mx}{m+1}\times\frac{m}{m}=\frac{m^2x}{m^2+m}$ and $\frac{(m-1)x}{m}\times\frac{m+1}{m+1}=\frac{m^2x-x}{m^2+m}$

$\therefore \frac{m^2x}{m^2+m}-\frac{m^2x-x}{m^2+m}=\frac{m^2x-m^2x+x}{m^2+m}=\frac{x}{m^2+m}.$

ART. 52.

(1.) $\frac{\cancel{3}a}{(2)\cancel{4}c}\times\frac{\cancel{2}a}{\cancel{3}c}=\frac{a^2}{2c^2}.$

(2.) $\frac{3b}{a+b}\times\frac{4a}{a-b}=\frac{12ab}{a^2-b^2}.$

(3.) $\frac{a}{a-b}\times\frac{a-b}{a+b}=\frac{(a+b)(a-b)}{a(a-b)}=\frac{a}{a+b}.$

(4.) $\frac{2a+b}{a^2-b^2}\times\frac{a-b}{a+b}=\frac{2a+b}{(a+b)(a-b)}\times\frac{a-b}{a+b}=\frac{2a+b}{(a+b)^2}.$

(5.) $\frac{a}{a+b}\times\frac{b}{a-b}\times\frac{a^2-b^2}{a^3-b^3}=\frac{ab(a^2-b^2)}{(a^2-b^2)(a^3-b^3)}=\frac{ab}{a^3-b^3}.$

(6.) $\frac{a}{\cancel{b}}\times\frac{\cancel{b}}{\cancel{c}}\times\frac{\cancel{c}}{\cancel{d}}\times\frac{\cancel{d}}{\cancel{e}}\times\frac{\cancel{e}}{f}=\frac{a}{f}$, by the note.

ART. 53.

(1.) $\frac{3a}{7x}\div\frac{2x}{7a}=\frac{3a}{\cancel{7}x}\times\frac{\cancel{7}a}{2x}=\frac{3a^2}{2x^2}.$

(2.) $\frac{ac}{bd}\div\frac{ac}{d}=\frac{ac}{bd}\times\frac{d}{ac}=\frac{acd}{acd\times b}=\frac{1}{b}.$

(3.) $\frac{a+b}{c}\div\frac{a-b}{c}=\frac{a+b}{\cancel{c}}\times\frac{\cancel{c}}{a-b}=\frac{a+b}{a-b}.$

(4) $\frac{x^2-4}{5}\div\frac{x+2}{4}=\frac{x^2-4}{5}\times\frac{4}{x+2}=\frac{(x+2)(x-2)\times 4}{5(x+2)}=\frac{4x-8}{5}.$

(5.) $\frac{4x+2}{3}\div\frac{2x+1}{5x}=\frac{2(2x+1)}{3}\times\frac{5x}{2x+1}=\frac{10x}{3}$.

(6.) $\frac{9x^2-3x}{5}\div\frac{x^2}{5}=\frac{3x(3x-1)}{\cancel{5}}\times\frac{\cancel{5}}{x^2}=\frac{9x-3}{x}$.

(7.) $\frac{a+b}{2}+\frac{a-b}{2}=\frac{a+b+a-b}{2}=\frac{2a}{2}=a$,

and $\frac{a+b}{2}-\frac{a-b}{2}=\frac{a+b-a+b}{2}=\frac{2b}{2}=b$.

(8.) $\frac{(a+x)^2}{2}=\frac{a^2+2ax+x^2}{2}$ and $\frac{(a-x)^2}{2}=\frac{a^2-2ax+x^2}{2}$

$\therefore\ \frac{a^2+2ax+x^2+a^2-2ax+x^2}{2}=\frac{2a^2+2x^2}{2}=a^2+x^2$, sum;

and changing the signs of the subtrahend, we have
$\frac{a^2+2ax+x^2-a^2+2ax-x^2}{2}=\frac{4ax}{2}=2ax$, difference.

(9.) $\frac{1}{x+w}$ and $\frac{1}{x-w}$ reduced to a common denominator become $\frac{x-w}{x^2-w^2}$ and $\frac{x+w}{x^2-w^2}$; hence

$\frac{1}{x+w}+\frac{1}{x-w}=\frac{x-w+x+w}{x^2-w^2}=\frac{2x}{x^2-w^2}$, sum;

and $\frac{1}{x-w}-\frac{1}{x+w}=\frac{x+w-x+w}{x^2-w^2}=\frac{2w}{x^2-w^2}$, difference.

(10.) $\frac{a+x}{a-x}\times\frac{a-x}{(a+x)(a-x)}=\frac{1}{a-x}$, the product;

$\frac{a+x}{a-x}\times\frac{(a+x)(a-x)}{(a-x)}=\frac{(a+x)^2}{a-x}$, the quotient.

(11.) Multiply both numerator and denominator of 1st by $1+ab$, and it becomes $\frac{a+a^2b+b-a}{1+ab-ab+a^2}=\frac{a^2b+b}{1+a^2}=\frac{(a^2+1)b}{1+a^2}=b$.

2d, $2\frac{x^2-\frac{1}{4}}{2x+1}+\frac{1}{2}=\frac{2x^2-\frac{1}{2}+x+\frac{1}{2}}{2x+1}$ by performing the multiplication by 2, and reducing to an improper fraction; and again cancelling the $\frac{1}{2}$ from the numerator, it becomes $\frac{2x^2+x}{2x+1}=\frac{(2x+1)x}{2x+1}=x$.

3d, $\frac{\frac{1}{a}+\frac{1}{ab^3}}{b-1+\frac{1}{b}}$, multiplying both numerator and denominator by ab^3, it becomes $\frac{b^3+1}{ab^4-ab^3+ab^2}=\frac{(b+1)(b^2-b+1)}{ab^2(b^2-b+1)}=\frac{b+1}{ab^2}$

4th, $\frac{x+y}{y}\times\frac{y^3-x^3}{y^2-x^2}=\frac{xy^2+y^3-x^3-x^2y}{y^3-x^2y}$

$$-\frac{2x}{y+x}\times\frac{y^2-yx}{y^2-yx}=\frac{-2xy^2+2x^2y}{y^3-x^2y}.$$

Collecting now the numerators of these two, and the third, which has the same denominator, we obtain

$$\frac{xy^2+y^3-x^3-x^2y-2xy^2+2x^2y+x^3-x^2y}{y^3-x^2y}=$$

$$\frac{y^3-xy^2}{y^3-x^2y}=\frac{y^2(y-x)}{y(y-x)(y+x)}=\frac{y}{y+x}.$$

Art. 55.

(1.) 2 raised to the fourth power is 16, and the exponents of each of the letters being multiplied by 4, gives $(2ax^2y^{\frac{1}{2}})^4=16a^4x^8y^2$.

(2.) $(7a^2px^3)^3=7\times7\times7a\quad p\quad x\quad=343a^6p^3x^9$.

(3.) $(\frac{2}{3}abc^2x)^5=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}(abc^2x)^5=\frac{32}{243}a^5b^5c^{10}x^5$.

(4.) $(\frac{1}{2}a^{\frac{1}{2}}b^{\frac{1}{4}}c)^8=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{16}\times\frac{1}{16}(a^{\frac{1}{2}}b^{\frac{1}{4}}c)^8=\frac{1}{256}a^4b^2c^8$.

(5.) $\left(\frac{7a^2bc}{xy^3}\right)^4=7\times7\times7\times7\left(\frac{a^2bc}{xy^3}\right)^4=\frac{2401a^8b^4c^4}{x^4y^{12}}$.

Art. 57.

(1.) Here the first term of the expansion will evidently be ,, ,, ,, a^4,

second, $a^4\times-\frac{4b}{a}$,, ,, $=-4a^3b$,

third, $-4a^3b\times-\frac{3b}{2a}$,, ,, $=6a^2b^2$,

fourth, $6a^2b^2\times-\frac{2b}{3a}$,, ,, $=-4ab^3$,

fifth, $-4ab^3\times-\frac{b}{4a}$,, ,, $=b^4$,

$\therefore (x-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

(2.) Here the first term will be $4a\times4a\times4a$, $=64a^3$,

second, $64a^3\times-\frac{3\times2x}{4a}$,, ,, $=-96a^2x$,

third, $-96a^2x\times-\frac{2\times2x}{2\times4a}$,, ,, $=\ 48ax^2$,

fourth, $48ax^2\times-\frac{2x}{3\times4a}$,, ,, $=-\ 8x^3$,

$\therefore (4a-2x)^3=64x^3-96a^2x+48ax^2-8x^3$.

(3.) The 9th power being the third power of the third, we *raise it to the third* by taking away the symbol of the cube

root, and then $(x+y)$ must be raised to third power; it will be the same as the third power of $(a+x)$, (Art. 57), only put x for a and y for x.

$$\therefore (\sqrt[3]{x+y})^9 = x^3+3x^2y+3xy^2+y^3.$$

(4.) First term, $2a\times 2a\times 2a\times 2a\times 2a \quad = \quad 32a^5,$

second term, $32a^5\times -\frac{5x}{2a} \quad " \quad = -80a^4x,$

third term, $-80a^4x\times -\frac{4x}{2\times 2a} \quad " \quad = \quad 80a^3x^2,$

fourth term, $80a^3x^2\times -\frac{3x}{3\times 2x} \quad " \quad = -40a^2x^3,$

fifth term, $-40a^2x^3\times -\frac{2x}{4\times 2x} \quad " \quad = \quad 10ax^4,$

sixth term $10ax^4\times -\frac{x}{5\times 2a} \quad " \quad = \quad -x^5,$

$$\therefore (2a-x)^5 = 32a^5-80a^4x+80a^3x^2-40a^2x^3+10ax^4-x^5.$$

(5.) $\{a-(x+y)\}^3$. Considering $x+y$ as one quantity, the expansion will be the same as the example, except in the letters; thus $a^3-3a^2(x+y)+3a(x+y)^2+(x+y)^3$.

Let now each term of this be expanded separately,

first term, $\quad " \quad " \quad " \quad a^3,$

second term, $-3a^2(x+y) \quad = -3a^2x-3a^2y,$

third term, $3a(x^2+2xy+y^2) = 3ax^2+6axy+3ay^2,$

fourth term, $-(x+y)^3 \quad = -x^3-3x^2y-3xy^2-y^3.$

and collecting all the terms, it becomes

$$a^3-x^3-y^3-3a^2x-3a^2y+3ax^2+3ay^2-3x^2y-3xy^2+6axy.$$

ART. 58.

(1.) The square root of 16 is 4, of a^2 is a, of b^4 is b^2, of c^6 is c^3; $\therefore \sqrt{16a^2b^4c^6} = 4ab^2c^3$.

(2.) The cube root of 64 is 4, of a^3 is a, of x^9 is x^3, and of y^6 is y^2.

$$\therefore \sqrt[3]{\frac{64a^3x^9}{y^6}} = \frac{4ax^3}{y^2}$$

(3.) The fourth root of a^2 is $a^{\frac{1}{2}}$, of x^4 is x, of y^6 is $y^{\frac{3}{2}}$, and of z^4 is z.

$$\therefore \sqrt[4]{\frac{a^2x^4y^6}{z^4}} = \frac{a^{\frac{1}{2}}xy^{\frac{3}{2}}}{z}.$$

(4.) The fifth root of 32 is 2, of x^{10} is x^2, of y^5 is y, and of v^{15} is v^3.

$$\therefore \sqrt[5]{\frac{32x^{10}y^5}{v^{15}}} = \frac{2x^2y}{v^3}.$$

(5.) The fourth root of x^2 is $x^{\frac{1}{2}}$, of y^4 is y, of z^8 is z^2.

$$\therefore \sqrt[4]{\frac{1}{x^2y^4z^8}} = \frac{1}{x^{\frac{1}{2}}yz^2}.$$

ART. 59.

(1.)

$$\begin{array}{l|l}
 & 9a^2+6ab+b^2(3a+b, \text{ root,} \\
 & 9a^2 \\
\hline
6a+b & +6ab+b^2 \\
 & +6ab+b^2 \\
\hline
\end{array}$$

(2.)

$$\begin{array}{l|l}
 & a^2+8ax+16x^2(a+4x, \text{ root,} \\
 & a^2 \\
\hline
2a+4x & 8ax+16x^2 \\
 & 8ax+16x^2 \\
\hline
\end{array}$$

(3.)

$$\begin{array}{l|l}
 & a^4c^2+4a^2cx+4x^2(a^2c+2x, \text{ root,} \\
 & a^4c^2 \\
\hline
2a^2c+2x & 4a^2cx+4x^2 \\
 & 4a^2cx+4x^2 \\
\hline
\end{array}$$

(4.)

$$\begin{array}{l|l}
 & 4a^2+12ab+9b^2+16ac+24bc+16c^2(2a+3b+4c, \text{ root,} \\
 & 4a^2 \\
\hline
4a+3b & 12ab+9b^2 \\
\quad 3b & 12ab+9b^2 \\
\hline
4a+6b+4c & 16ac+24bc+16c^2 \\
 & 16ac+24bc+16c^2 \\
\hline
\end{array}$$

(5.)

$$\begin{array}{l|l}
 & x^6+4x^5+2x^4+9x^2-4x+4(x^3+2x^2-x+2, \text{ root,} \\
 & x^6 \\
\hline
2x^3+2x^2 & 4x^5+2x^4 \\
\quad 2x^2 & 4x^5+4x^4 \\
\hline
2x^3+4x^2-x & -2x^4 \qquad +9x^2 \\
\quad -x & -2x^4-4x^3+x^2 \\
\hline
2x^3+4x^2-2x+2 & 4x^3+8x^2-4x+4 \\
 & 4x^3+8x^2-4x+4 \\
\hline
\end{array}$$

ART. 60.

(1.)

$$\begin{array}{l|l}
 & 27a^3-27a^2x+9ax^2-x^3(3a-x, \text{ root,} \\
 & 27a^3 \\
\hline
27a^2-9ax+x^2 & -27a^2x+9ax^2-x^3 \\
 & -27a^2x+9ax^2-x^3 \\
\hline
\end{array}$$

(2.)

$$\begin{array}{l|l}
 & a^6+6a^4b^2+12a^2b^4+8b^6(a^2+2b^2, \text{ root,} \\
 & a^6 \\
\hline
3a^4+6a^2b^2+4b^4 & 6a^4b^2+12a^2b^4+8b^6 \\
 & 6a^4b^2+12a^2b^4+8b^6 \\
\hline
\end{array}$$

(3.) $x^3-9x^2y+27xy^2-27y^3(x-3y$, root,

x^3

$3x^2-9xy+9y^2 \,|\, -9x^2y+27xy^2-27y^3$

$-9x^2y+27xy^2-27y^3$

SIMPLE EQUATIONS.

(1.) $x-6+25=36$,
$x=36+6-25$, by transposition,
$\therefore x=17$.

(2.) $3x-5=23-x$,
$3x+x=23+5$, by transposition,
$4x=28$, by collecting the terms,
$\therefore x=7$, by Rule 2d.

(3.) $7x-3=5x+13$,
$7x-5x=13+3$, by transposition,
$2x=16$, by collecting the terms,
$\therefore x=8$, by Rule 2d.

(4.) $3x+5=10x-16$,
$5+16=10x-3x$, by transposition,
$21=7x$, by collecting the terms,
$\therefore 3=x$, by dividing by 7.

(5.) $2x+11=7x-14$,
$11+14=7x-2x$, by transposition,
$25=5x$, by collecting the terms,
$\therefore 5=x$, by dividing by 5.

(6.) $15x-24=20+\frac{1}{3}x$,
$45x-72=60+x$, by multiplying by 3,
$45x-x=60+72$, by transposition,
$44x=132$, by collecting the terms,
$\therefore x=3$, by dividing by 44.

(7.) $x+\frac{x}{2}-\frac{x}{3}=4x-17$,

$6x+3x-2x=24x-102$, by multiplying by 6,
$6x+3x-2x-24x=-102$, by transposition,
$-17x=-102$, by collecting the terms,
$\therefore x=6$, by dividing by -17.

(8. $(x-\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=7$,

$12x-6x+4x-3x=84$, by multiplying by 12,
$7x=84$, by collecting the x's,
$\therefore x=12$, by dividing by 7.

(9.) $\frac{x}{3}+\frac{x}{4}+\frac{x}{6}+\frac{x}{8}+\frac{x}{12}=46$,

$8x+6x+4x+3x+2x=46\times 24$, by mult. by 24,

$23x=46\times 24$, by collecting,

$\therefore x=48$, by dividing by 23.

(10.) $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}$,

$6x+4x-3x=6$, by multiplying by 12,

$7x=6$, by collecting the terms,

$\therefore x=\frac{6}{7}$, by dividing by 7.

(11.) $\frac{x+1}{2}+\frac{x+2}{3}=14+\frac{5-x}{4}$,

$6x+6+4x+8=168+15-3x$, by mult. by 12.

$6x+4x+3x=168-6-8+15$, by transposition,

$13x=169$, by collecting the terms,

$\therefore x=13$, by dividing by 13.

(12.) $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{5x+1}{4}$,

$6x+6+4x+8=192-15x-3$, by mult. by 12.

$6x+4x+15x=192-6-8-3$, by transposition,

$25x=175$, by collecting the terms,

$\therefore x=7$, by dividing by 25.

(13.) $x-\frac{x-2}{3}=5\frac{3}{4}-\frac{x+10}{5}+\frac{x}{4}$,

$60x-20x+40=345-12x-120+15x$, mult. by 60,

$60x-20x+12x-15x=345-120-40$, by transp.

$37x=185$, by collecting the terms,

$\therefore x=5$, by dividing by 37.

(14.) $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$,

$6x+9x-15=72-4x+8$, by multiplying by 6,

$6x+9x+4x=72+15+8$, by transposition,

$19x=95$, by collecting the terms,

$\therefore x=5$, by dividing by 19.

(15.) $\frac{1}{a}+\frac{b}{x}=c$,

$x+ab=acx$, by multiplying by ax,

$ab=acx-x$, by transposition,

$ab=(ac-1)x$, by collecting the terms,

$\therefore \frac{ab}{ac-1}=x$, by dividing by $(ac-1)$.

(16.) $ax+b^2=bx+a^2$,

$(a-b)x=a^2-b^2$, by transposing and collecting,

$\therefore x=a+b$, by dividing by $(a-b)$.

(17.) $bx+2x-a=3x+2c$,
$bx+2x-3x=a+2c$, by transposition,
$(b-1)x=a+2c$, by collecting,
$\therefore x=\frac{a+2c}{b-1}$, by dividing by $(b-1)$.

(18.) $3x-\frac{a}{b}+cx=\frac{a+x}{3}-\frac{b-x}{a}$,
$9abx-3a^2+3abcx=a^2b+abx-3b^2+3bx$,
by multiplying both sides by $3ab$,
$(9ab+3abc-ab-3b)x=3a^2+a^2b-3b^2$, by transp.
or $(8ab+3abc-3b)x=3a^2+a^2b-3b^2$,

$$\therefore x=\frac{3a^2+a^2b-3b^2}{8ab+3abc-3b}.$$

19. Let the number sought be represented by x, then by the question, $x+\frac{x}{2}+\frac{x}{3}=154$,

and mult. by 6, $6x+3x+2x=924$, Rule 3d,
hence collecting and dividing, $x=84$.

20. Let $x=$ the number sought; then by the question,

$$x+\frac{x}{3}+\frac{x}{4}-\frac{x}{6}=17,$$

and mult. by 12, $12x+4x+3x-2x=204$,
collecting the terms, $17x=204$,
$\therefore x=12$.

21. Let $x=$ those who voted for the successful candidate; then the majority will be (Schol. Art. 70), $(2x-311)$, which by the question is 79; hence

$$2x-311=79,$$
$$2x=390,$$
$$\therefore x=195,$$
and $x-79=195-79=116$.

NOTE. The number that voted for the unsuccessful candidate may also be represented by x, then $x+79$ will be the number who voted for the successful candidate, and $2x+79=311$; hence $x=116$ as before. It may also be easily solved by making those who voted for the successful candidate x, and those who voted for the other y, then $x+y=311$, and $x-y=79$; for the solution of which see the Rules for Simultaneous Equations.

22. Let x represent the number; then by the question,

$$x-50=\frac{x}{2}+\frac{x}{4}+\frac{x}{6},$$

Mult. by 12, $12x-600=6x+3x+2x$,
Transposing and collecting, $x=600$.

23. Let x *represent the* wife's age at the time of the *marriage; then* $3x$ *will* represent the husband's age at the

same time. Fifteen years after the marriage the wife would be $x+15$, and the husband $3x+15$.

Hence by the question, $3x+15=2(x+15)$,
or $3x+15=2x+30$;
$\therefore x=15$ by transposing,
and $3x=45$.

24. Let x represent B's share, then since A had twice as much, $2x$ will represent A's share, and since C had as much as A and B together, C's share will be represented by $3x$; but the sum of these shares is L.300.

Hence $x+2x+3x=300$, $\therefore$ by collecting and dividing, $x=50$ B's share, and $2x=100$ A's share, and $3x=150$ C's share.

25. Let $x=$ the number of hours required; then since the first could fill the cistern in 12 hours, it could fill $\frac{1}{12}$ in one hour, and consequently $\frac{x}{12}$ in x hours; in the same manner the second could fill $\frac{x}{8}$ in x hours; but in x hours by supposition the cistern was filled, or the sum of these fractions is $=1$.

Hence $\frac{x}{12}+\frac{x}{8}=1$, multiplying this by 96, we have
$20x=96$,
$\therefore x=4\frac{4}{5}$.

Note. If the time in which the first could fill the cistern had been represented by a, and the time in which the second could fill it by b, then we would have $x=\frac{ab}{a+b}$: Or, *The times being given in which each of two agents acting separately can produce a given effect, the time in which they can jointly produce the same effect, is the product of the times divided by the sum of the times.*

26. Let x represent the number of yards at 7s.; then $40-x$ will represent the number at 3s.; then by the question, $7x=3(40-x)$,
hence $10x=120$,
and $\therefore x=12$ yards at 7s.,
hence also $40-12=28$ yards at 3s.

27. Let x represent B's money; then $3x$ will be A's, and when B gives A L.50, A will have $3x+50$, and B will have $x-50$; but by the question the first will be four times the second. Hence $3x+50=4(x-50)$,
or, $3x+50=4x-200$,
$250=x$ B's money,
$\therefore 3x=750$ A's money.

28. Let $x =$ the number of gallons that each cask contained; then $x-34$ is what remained in the first, and $x-80$ what remained in the second, and by the question, the first remainder was three times the second, that is,

$$x-34=3(x-80),$$
$$\text{or } x-34=3x-240,$$
$$\therefore x=103.$$

29. Let $x =$ the number of crowns; then $202-x$ will be the number of half-guineas, $10x$ will be the number of sixpences paid in crowns, and $21(202-x)$ the number of sixpences paid in half-guineas.

$$\text{Hence } 4000=10x+21(202-x),$$
$$\text{or } 4000=10x+4242-21x,$$
$$\text{By transp. } 11x=242,$$
$$\therefore x=22, \text{ the number of crowns,}$$
$$\text{and } 202-22=180, \text{ the number of half-guineas.}$$

30 and 31. Let $x =$ the time sought; then (31) $\frac{x}{a}$ is the quantity filled by the first pipe, $\frac{x}{b}$ that filled by the second, $\frac{x}{c}$ that filled by the third; and since these quantities together filled the cistern,

$$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1, \text{ multiplying by } abc,$$
$$\text{we have } (bc+ac+ab)x=abc,$$
$$\therefore x=\frac{abc}{ab+ac+bc}.$$

This shows that if the time be given in which each of three agents acting separately will produce a given effect, the time in which they will produce the same effect acting jointly, is *the product of the three times divided by the sum of the products of every pair of the times.*

For Question 30th, we have by substitution of the proper values of a, b, c,

$$x=\frac{8\times16\times10}{128+80+160}=\frac{1280}{368}=$$
$$3 \text{ hours } 28\tfrac{16}{23} \text{ minutes.}$$

SIMULTANEOUS EQUATIONS.

1. By Rule 1st.	1 2	$3x+2y=56$ $2x+3y=54$	Given equations.
(1)×2	3	$6x+4y=112.$	
(2)×3	4	$6x+9y=162.$	
(4)—(3)	5	$5y=50.$	
(5)÷5	6	$\therefore y=10$, value of y.	

In the same manner, if the first be multiplied by 3, and the second by 2, and the products subtracted, we will get a value of x; or the value of x may be found from either equation, by substituting the value of y already found, which will reduce it to an equation with one unknown quantity whose value may easily be found.

2. By Rule 2d, $5x-7y=8, \therefore x=\frac{7y+8}{5}$,

$$4x-y=34, \therefore x=\frac{34+y}{4}.$$

Hence also, $\frac{7y+8}{5}=\frac{34+y}{4}$, since each is $=x$,

$28y+32=170+5y$. By mult. by 20,
$23y=138$,
$\therefore y=6$, and $x=\frac{34+6}{4}=10$.

3.		
By Rule 3d.	1	$\frac{1}{3}x+\frac{1}{2}y=7$ Given equations.
	2	$\frac{1}{9}x+\ y=9$
(2) transp.	3	$y=9-\frac{1}{9}x$.
	4	$\frac{1}{3}x+\frac{1}{2}(9-\frac{1}{9}x)=7$, by subst. in (1).
(4) × 18	5	$6x+81-x=126$.
	6	$5x=45$, by transposition, &c.
(6) ÷ 5	7	$\therefore x=9$, value of x.

and $y=9-\frac{1}{9}x=9-1=8$, value of y.

4.	1	$3x+\frac{1}{2}y=38$ Given equations.
By Rule 4th.	2	$\frac{1}{3}x+2y=12$
(2) × n	3	$\frac{1}{3}nx+2ny=12n$.
(3) + (1)	4	$(\frac{1}{3}n+3)x+(2n+\frac{1}{2})y=12n+38$.
Let coeff. of $y=0$	5	$(\frac{1}{3}n+3)x=12n+38$.
which gives	6	$2n+\frac{1}{2}=0$.
	7	$\therefore n=-\frac{1}{4}$, subst. this value
in (5) and we have	8	$2\frac{11}{12}x=35$.
	9	$35x=35\times 12$.
	10	$\therefore x=12$, value of x.
Next let the coeff.	11	$(2n+\frac{1}{2})y=12n+38$.
of $x=0$, which gives	12	$\frac{1}{3}n+3=0$.
	13	$\therefore n=-9$, subst. this value
in (11) and we have	14	$-17\frac{1}{2}y=-70$.
	15	$\therefore y=4$, value of y.

The above equations solved by Rule 1st would be as follows:—

(1) × 2	3	$6x+y=76$.
(2) × 18	4	$6x+36y=216$.
(4)—(3)	*5*	$35y=140$.
(5)÷35	*6*	$\therefore y=4$. value of y.

(1)×4	7	$12x+2y=152$.	
(7)—(2)	8	$11\frac{2}{3}x=140$.	
(8)×3	9	$35x=420$.	
(9)÷35	10	$\therefore x=12$, value of x.	

5.	1	$x^2-y^2=20$	Given equations.
	2	$x+y=10$	
(1)÷(2)	3	$x-y=2$.	
(2)+(3)	4	$2x=12$.	
(4)÷2	5	$\therefore x=6$, value of x.	
(2)—(3)	6	$2y=8$.	
(6)÷2	7	$\therefore y=4$, value of y.	

6.	1	$x+y=s$	Given equations.
	2	$x-y=d$	
(1)+(2)	3	$2x=s+d$.	
(3)÷2	4	$\therefore x=\frac{1}{2}(s+d)$, value of x.	
(1)—(2)	5	$2y=s-d$.	
(5)÷2	6	$y=\frac{1}{2}(s-d)$, value of y.	

The solution of this question proves a theorem of considerable use in several parts of mathematics, viz. *If there be two unequal quantities whose sum and difference are known; the greater is equal to half the sum together with half the difference; and the less is equal to half the sum diminished by half the difference.*

Question 21, page 42, may at once be solved by the application of this principle.

7.	1	$\frac{3x+2y}{6}+2x=16$	Given equations.
	2	$\frac{2x-3}{9}+\frac{2x+7y}{11}=4$	
(1)×6	3	$3x+2y+12x=96$.	
	4	$15x+2y=96$, collecting the terms.	
(2)×99	5	$22x-33+18x+63y=396$.	
	6	$40x+63y=429$, collecting and transp.	
(4)×8	7	$120x+16y=768$.	
(6)×3	8	$120x+189y=1287$.	
(8)—(7)	9	$173y=519$.	
(9)÷173	10	$\therefore y=3$, value of y.	
(4)×63	11	$945x+126y=6048$.	
(6)×2	12	$80x+126y=858$.	
(11)—(12)	13	$865x=5190$.	
(13)÷865	14	$\therefore x=6$, value of x.	

8.	1	$ax+by=c$	Given equations.
By Rule 1st.	2	$a'x+b'y=c'$	
(1)×a′	*3*	$aa'x+a'by=a'c$.	

$(2)\times a$	4	$aa'x+ab'y=ac'$.
$(3)-(4)$	5	$(a'b-ab')y=a'c-ac'$.
$(5)\div(a'b-ab')$	6	$\therefore y=\frac{a'c-ac'}{a'b-ab'}$, value of y.
$(1)\times b'$	7	$ab'x+bb'y=b'c$.
$(2)\times b$	8	$a'bx+bb'y=bc'$.
$(7)-(8)$	9	$(ab'-a'b)x=b'c-bc'$.
	10	$\therefore x=\frac{b'c-bc'}{ab'-a'b}$, value of x.

NOTE. If the (7) had been subtracted from the (8), the value of x would have been $\frac{bc'-b'c}{a'b-ab'}$; and in the same manner if the (3) had been subtracted from the (4), the value of y would have been found $\frac{ac'-a'c}{ab'-a'b}$. These two sets of expressions, although differing in appearance, are the same in reality, and may be found the one from the other by multiplying both terms of the fraction by -1, or they may be proved identical by substituting the same numerical values of a, b, c, a', b', and c', in both forms, and the same values for x and y will be obtained.

By Rule 2d.

From (1)$x=\frac{c-by}{a}$, and from (2)$x=\frac{c'-b'y}{a'}$,

Hence $\frac{c-by}{a}=\frac{c'-b'y}{a'}$,

$a'c-a'by=ac'-ab'y$, by multiplying by aa',
$(ab'-a'b)y=ac'-a'c$, by transposing,
$\therefore y=\frac{ac'-a'c}{ab'-a'b}$, value of y; and in the same manner may the value of x be found.

The calculation for finding the values of x and y by the third Rule would be nearly the same as by the second.

By the 4th Rule.

$(1)\times n$	11	$nax+nby=nc$.
$(11)+(2)$	12	$(na+a')x+(nb+b')y=nc+c'$.
If $nb+b'=0$,	13	$(na+a')x=nc+c'$.
then $n=-\frac{b'}{b}$,	14	$\left(a'-\frac{ab'}{b}\right)x=c'-\frac{cb'}{b}$ $\begin{cases}\text{by substituting}\\\text{for } n \text{ its value.}\end{cases}$
	15	$(a'b-ab')x=bc'-b'c$, by mult. by b.
	16	$\therefore x=\frac{bc'-b'c}{a'b-ab'}$, value of x.
If $na+a'=0$,	17	$(nb+b')y=nc+c'$.
then $n=-\frac{a'}{a}$,	18	$\left(b'-\frac{ba'}{a}\right)y=c'-\frac{a'c}{a}$ $\begin{cases}\text{by substituting}\\\text{the value of } n.\end{cases}$
	19	$(ab'-a'b)y=ac'-a'c$, by mult. by a.
	20	$\therefore y=\frac{ac'-a'c}{ab'-a'b}$, value of y.

The above solutions have given $x=\frac{bc'-b'c}{a'b-ab'}$ and $\frac{b'c-bc'}{ab'-a'b}$, also $y=\frac{ac'-a'c}{ab'-a'b}$ and $\frac{a'c-ac'}{a'b-ab'}$; these are not two values, but the same value in two different forms.

PAGE 47, Ex. 1.

	1	$x+y-z=10$	The given equations.
	2	$x-y+z=6$	
	3	$y+z-x=2$	
(1)+(2)+(3)	4	$x+y+z=18.$	
(4)—(1)	5	$2z=8.$	
	6	$\therefore z=4$, value of z.	
(4)—(2)	7	$2y=12.$	
	8	$\therefore y=6$, value of y.	
(4)—(3)	9	$2x=16.$	
	10	$\therefore x=8$, value of x.	

Otherwise,

(1)+(2)	11	$2x=16.$
	12	$\therefore x=8$, value of x.
(1)+(3)	13	$2y=12.$
	14	$\therefore y=6$, value of y.
(2)+(3)	15	$2z=8.$
	16	$\therefore z=4$, value of z.

2.	1	$x+y+z=35$	The given equations.
By Rule 1st.	2	$x+2y+3z=66$	
	3	$\frac{1}{2}x+\frac{1}{3}y\times\frac{1}{4}z=13$	
(3)×2	4	$x+\frac{2}{3}y+\frac{1}{2}z=26.$	
(2)—(1)	5	$y+2z=31.$	
(1)—(4)	6	$\frac{1}{3}y+\frac{1}{2}z=9.$	
(6)×(3)	7	$y+\frac{3}{2}z=27.$	
(5)—(7)	8	$\frac{1}{2}z=4.$	
(8)×2	9	$\therefore z=8$, value of z.	
Substituting in (5)	10	$y+16=31.$	
Transp. (10)	11	$\therefore y=15$, value of y.	
(9)+(11)	12	$y+z=23.$	
(1)—(12)	13	$x=12$, value of x.	

3.	1	$x+y=28$	The given equations.
	2	$x+z=26$	
	3	$y+z=22$	
(1)+(2)+(3)	4	$2x+2y+2z=76.$	
(4)÷2	5	$x+y+z=38.$	
(5)—(3)	6	$x=16$, value of x.	
(5)—(2)	7	$y=12$, value of y.	
(5)—(1)	8	$z=10$, value of z.	

Otherwise,

(1)—(2)	9	$y-z=2.$	
(9)+(3)	10	$2y=24.$	
(10)÷2	11	$\therefore y=12$, value of y.	
(2)—(3)	12	$x-y=4.$	
(12)+(1)	13	$2x=32.$	
(13)÷2	14	$\therefore x=16$, value of x.	
(1)—(3)	15	$x-z=6.$	
(2)—(15)	16	$2z=20.$	
(16)÷2	17	$z=10$, value of z.	

4.	1	$x+y+z=90$	The given equations.
	2	$2x-20=3y-40$	
	3	$x+20=2z+5$	
Transp. (2)	4	$2x-3y=-20.$	
Transp. (3)	5	$x-2z=-15.$	
(1)×2	6	$2x+2y+2z=180.$	
(5)×2	7	$2x-4z=-30.$	
(6)—(4)	8	$5y+2z=200.$	
(6)—(7)	9	$2y+6z=210.$	
(9)×2½	10	$5y+15z=525.$	
(10)—(8)	11	$13z=325.$	
(11)÷13	12	$\therefore z=25$, value of z.	
(8)×3	13	$15y+6z=600.$	
(13)—(9)	14	$13y=390.$	
(14)÷13	15	$\therefore y=30$, value of y.	
(12)+(15)	16	$y+z=55.$	
(1)—(16)	17	$x=35$, value of x.	

SOLUTION OF PROBLEMS PRODUCING EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

1. Let $x=$ the first, and $y=$ the second;

then	1	$x+2y=21,$	by the question.
	2	$2x+\frac{1}{2}y=14,$	
(1)×2	3	$2x+4y=42,$	
(3)—(2)	4	$3\frac{1}{2}y=28,$	
(4)÷3½	5	$y=8$, value of the second.	
(2)×4	6	$8x+2y=56,$	
(6)—(1)	7	$7x=35,$	
(7)÷7	8	$x=5$, value of the first.	

2. Let $x=$ the first, and $y=$ the second;

then	1	$\frac{1}{2}x+\frac{1}{3}y=7,$	by the question.
	2	$\frac{1}{3}x+\frac{1}{4}y=5,$	
(1)×2	3	$x+\frac{2}{3}y=14,$	
(2)×3	4	$x+\frac{3}{4}y=15,$	

(4) $-$(3)	5	$\frac{1}{12}y=1$,
(5) $\times 12$	6	$y=12$, value of the second.
(1) $\times 3$	7	$\frac{3}{2}x+y=21$,
(2) $\times 4$	8	$\frac{4}{3}x+y=20$,
(7) $-$(8)	9	$\frac{1}{6}x\quad=1$,
(9) $\times 6$	10	$x\quad=6$, value of the first.

3. Let $x=$the first and $y=$ the second;

then	1	$x+5=2y$,
	2	$3x\quad=4y+3$,

$$\left.\begin{array}{l} x+5=2y, \\ 3x\quad=4y+3, \end{array}\right\}\text{by the question.}$$

From (1) we have $x=2y-5$, and from (2) we have $x=\frac{4}{3}y+1$;

$\therefore 2y-\ 5=\frac{4}{3}y+1$, by Rule 2,
and $6y-15=4y+3$, by mult. by 3,
$2y=18$, by transp.
$\therefore y=9$, value of y.
$x=\frac{4}{3}y+1=13$, value of x.

4. Let the fraction be represented by $\frac{x}{y}$; then

$$\left.\begin{array}{l} \dfrac{x+1}{y-2}=\frac{1}{2}, \\ \dfrac{x+3}{y-2}=\frac{3}{5}, \end{array}\right\}\text{by the question.}$$

From the first, $(y-2)=2x+2$, and from the second, $(y-2)=\frac{5}{3}x+5$;

$\therefore 2x+2=\frac{5}{3}x+5$, by Rule 2,
and $6x+6=5x+15$, by mult. by 3;
$\therefore x=9$, the numerator;
Hence $y-2=2x+2=18+2=20$;
$\therefore y=22$, the denominator;

$$\text{Therefore}\ \frac{x}{y}=\tfrac{9}{22}.$$

5. Let the left hand digit be represented by x, and the right hand digit by y, then the number will be represented by $10x+y$; hence

	1	$x+y=\dfrac{10x+y}{4}$, by the question.
	2	$xy=\dfrac{10x+y}{2}$,
(1) $\times 4$	3	$4x+4y=10x+y$.
Transp.	4	$3y=6x$.
(4) $\div 3$	5	$y=2x$.
Substituting in (2)	6	$2x^2=5x+x=6x$.
(6) $\div 2x$	7	$x=3$, the digits sought;
Substituting in (5)	8	$y=6$, hence the number is 36.

6. Let the notation be the same as in last example;

then	1	$10x+y=6y,$	by the quest.
	2	$10x+y+27=10y+x,$	
Transposing (1)	3	$10x-5y=0.$	
Transposing (2)	4	$9x-9y=-27.$	
$(3)\times 9$	5	$90x-45y=0.$	
$(4)\times 10$	6	$90x-90y=-270.$	
$(5)-(6)$	7	$45y=270,$	
$(7)\div 45$	8	$y=6,$ value of unit's figure.	
Substituting in (1)	9	$10x+6=36.$	
Transposing (9)	10	$10x=30.$	
$(10)\div 10$	11	$x=3,$ value of left hand digit.	

Hence the number sought is 36.

NOTE. The above equation may also be solved with less labour, by finding a value of y in terms of x, from the first equation, and substituting this value instead of it in the second, (Rule 3.) Thus, from the first $y=2x$, substituting this value of y, in the second, gives, after transposition, $9x=27$, therefore $x=3$, consequently $y=6$, and the number sought is $\therefore$ 36.

7. Let the digits be represented by x, y, and z, respectively;

then	1	$x+y+z=7,$	By the question.
	2	$2x+2z=5y,$	
	3	$100x+10y+z-297=100z+10y+x,$	
Transposing (3)	4	$99x-99z=297,$	
$(1)\times 2$	5	$2x+2y+2z=14,$	
$(5)-(2)$	6	$2y=14-5y.$	
Transp. and dividing (6)	7	$y=2.$	
$(2)\times 49\frac{1}{2}$	8	$99x+99z=495,$ substituting value of y.	
$(4)+(8)$	9	$198x=792.$	
$(9)\div 198$	10	$x=4.$	
$(8)-(4)$	11	$198z=198.$	
$(11)\div 198$	12	$z=1.$	

Hence the number sought is 421.

8. Let the first be x, the second y, and the third z; then

	1	$x+\frac{1}{2}y+z\frac{1}{2}=34$	By the question.
	2	$\frac{1}{3}x+y+\frac{1}{3}z=34$	
	3	$\frac{1}{4}x+\frac{1}{4}y+z=34$	
$(2)\times 3$	4	$x+3y+z=102.$	
$(3)\times 4$	5	$x+y+4z=136.$	
$(4)-(1)$	6	$2\frac{1}{2}y+\frac{1}{2}z=68.$	
$(5)-(1)$	7	$\frac{1}{2}y+3\frac{1}{2}z=102.$	

(6)×2	8	$5y+z=136.$
(7)×10	9	$5y+35z=1020.$
(9)—8	10	$34z=884.$
(10)÷34	11	$\therefore z=26$, value of third.
Substituting in (8)	12	$5y+26=136.$
Transp. and divid.	13	$y=22$, value of second.
Substituting in (1)	14	$x+11+13=34.$
Transposing (14)	15	$x=10$. value of first.

9. Since the digits are in arithmetical progression, if the middle digit be represented by y, and the common difference by x, the others can be represented by $y+x$ and $y-x$; and since by subtracting from the number, the digits were inverted, the greatest digit must be in the place of hundreds, and the least in the unit's place; therefore we have the following equations:—

	1	$\frac{100(y+x)+10y+y-x}{3y}=48.$
	2	$\left\{\begin{array}{l}100(y+x)+10y+y-x-198= \\ \quad 100(y-x)+10y+y+x.\end{array}\right.$
(1) reduced	3	$111y+99x=144y.$
Transp. and div. (3)	4	$3x=y.$
Collecting (2)	5	$111y+99x-198=111y-99x.$
Transposing (5)	6	$198x=198.$
(6)÷198	7	$x=1.$
Subst. in (4), $x=1$	8	$y=3.$

Therefore the first figure being $(y+x)=4$, and the third being $(y-x)=2$, and the number sought is 432.

10. Let x = the time in which A could finish the whole, y = the time in which B could do it, and z the time in which C could do it. In order to obtain a general solution of all questions similar to this, put a for 8 days, b for 9 days, and c for 10 days; then since A could do it in x days, he could do $\frac{1}{x}$ in a day, and since A and B together could do it in a days, they could do $\frac{1}{a}$ in a day, and so of the others; hence

	1	$\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$	By the question.
	2	$\frac{1}{x}+\frac{1}{z}=\frac{1}{b}$	
	3	$\frac{1}{y}+\frac{1}{z}=\frac{1}{c}$	
(1)+(2)+(3)	4	$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$	

$(4)\div 2$	5	$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}.$
$(5)-(3)$	6	$\frac{1}{x}=\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2c}.$
$(6)\times 2abcx$	7	$2abc=(bc+ac-ab)x.$
$(7)\div(ac+bc-ab)$	8	$x=\frac{2abc}{(ac+bc-ab)}$, value of x.
$(5)-(2)$	9	$\frac{1}{y}=\frac{1}{2a}+\frac{1}{2c}-\frac{1}{2b}.$
$(9)\times 2abcy$	10	$2abc=(bc+ab-ac)y.$
$(10)\div(bc+ab-ac)$	11	$y=\frac{2abc}{(bc+ab-ac)}$, value of y.
$(5)-(1)$	12	$\frac{1}{z}=\frac{1}{2b}+\frac{1}{2c}-\frac{1}{2a}.$
$(12)\times 2abcz$	13	$2abc=(ac+ab-bc)z.$
$(13)\div(ac+ab-bc)$	14	$z=\frac{2abc}{(ac+ab-bc)}$, value of z.

Substituting now in these values of x, y, and z, the values of a, b, and c, the times in which A, B, and C could perform the work when acting alone, will be found thus:—

$$x=\frac{2abc}{ac+bc-ab}=\frac{1440}{98}=\frac{720}{49}=14\tfrac{34}{49}.$$

$$y=\frac{2abc}{bc+ab-ac}=\frac{1440}{82}=\frac{720}{41}=17\tfrac{23}{41}.$$

$$z=\frac{2abc}{ac+ab-bc}=\frac{1440}{62}=\frac{720}{31}=23\tfrac{7}{31}.$$

SOLUTION OF QUADRATIC EQUATIONS.

1. Given $x^2=4x+45$.

1	$x^2-4x=45$, by transposition.
2	$x^2-4x+4=49$, completing square, (Art. 82.)
3	$x-2=\pm 7$, extracting root.
4	$\therefore x=2\pm 7=9$, or -5.

It may also be solved at once, by substituting in the general formula, (Art. 80), the proper values of a, b, and c, found in (1) above, which gives

$$x=\frac{4\pm\sqrt{16+180}}{2}=\frac{4\pm 14}{2}=9,\text{ or }-5.$$

2. Given	1	$5x^2+7+4x=35.$
Transp.	2	$5x^2+4x=28.$
$(2)\times 20+4^2$	3	$100x^2+80x+16=576.$
Extract root of (3)	4	$10x+4=\pm 24.$
By transp. (4) &c.	5	$\therefore x=\frac{\pm 24-4}{10}=2$, or $-2\frac{4}{5}$.

3. Given	1	$x^2-50=5-6x.$
	2	$x^2+6x=55.$
Art. 82.	3	$x^2+6x+9=64.$
Ex. root of (3)	4	$x+3=\pm 8.$
Transp.	5	$x=\pm 8-3=5$, or $-11.$

4. Given	1	$4x-14=\frac{14-x}{x+1}.$
(1) × ($x+1$)	2	$4x^2+4x-14x-14=14-x.$
Transp. (2) &c.	3	$4x^2-9x=28.$

∴ Formula (Art. 80), $x=\frac{9\pm\sqrt{81+448}}{8}=\frac{9\pm 23}{8}$

$=4$, or $-\frac{7}{4}.$

5. Given	1	$3x+4=2+\frac{1121}{x}.$
Transp. (1) and	2	$3x^2+2x=1121$, mult. by x.
Art. 81	3	$36x^2+24x+4=13456.$
Extracting root	4	$6x+2=\pm 116.$
Transp. (4) and dividing by 6.	5	$\therefore x=\frac{-2\pm 116}{6}=19$, or $-19\frac{2}{3}.$

6. Given	1	$4-\frac{2x-11}{x-3}=\frac{4x-2}{6}=\frac{2x-1}{3}.$
Mult. (1) by $3(x-3)$	2	$12x-36-6x+33=2x^2-7x+3.$
Transp. (2)	3	$2x^2-13x=-6.$
Art. 81	4	$16x^2-104x+169=121.$
Extracting root	5	$4x-13=\pm 11.$
	6	$\therefore x=\frac{13\pm 11}{4}=6$, or $\frac{1}{2}.$

7. Given	1	$\frac{1}{2}x^2=\frac{1}{3}x+\frac{5}{8}.$
(1) reduced	2	$12x^2-8x=15.$
Art. 81	3	$576x^2-384x+64=784.$
Extracting root	4	$24x-8=\pm 28.$
	5	$\therefore x=\frac{8\pm 28}{24}=1\frac{1}{2}$, or $-\frac{5}{6}.$

8. Given	1	$2x^2+2x=24$, by subtracting $6\frac{1}{2}$ from both.
Art. 81	2	$16x^2+16x+4=196.$
Extracting root	3	$4x+2=\pm 14.$
	4	$\therefore x=\frac{-2\pm 14}{4}=3$, or $-4.$

9. Given	1	$x-1=\frac{110}{x}.$
(1) × x	2	$x^2-x=110.$

Art. 82	3	$x^2-x+\frac{1}{4}=\frac{441}{4}$.
Extracting root	4	$x-\frac{1}{2}=\pm\frac{21}{2}$.
	5	$\therefore x=\frac{1}{2}\pm\frac{21}{2}=11$, or -10.

10. Given	1	$\frac{1}{5}x^2=24x-700$.
Transp. (1) and (×5)	2	$x^2-120x=-3500$.
Art. 82	3	$x^2-120x+3600=100$.
Extracting root	4	$x-60=\pm10$.
	5	$\therefore x=60\pm10=70$, or 50.

11. Given	1	$\frac{30}{x}-\frac{30}{x+1}=\frac{9}{x+2}$.
(1) × $x(x+1)(x+2)$	2	$30x^2+90x+60-30x^2-60x=9x^2+9x$.
(2) Transposed	3	$9x^2-21x=+60$.
(3) ÷ 3	4	$3x^2-7x=+20$.
Art. 81	5	$36x^2-84x+49=289$.
Extracting root	6	$6x-7=\pm17$.
	7	$\therefore x=\frac{7\pm17}{6}=4$, or $-1\frac{2}{3}=-\frac{5}{3}$.

(12.) Given	1	$\frac{1}{x-3}+\frac{1}{x+4}=1\frac{1}{8}$.
(1) × $8(x-3)(x+4)$	2	$8x+32+8x-24=9x^2+9x-108$.
(2) Transposed	3	$9x^2-7x=116$.
Art. 81.	4	$324x^2-252x+49=4225$.
Extracting root	5	$18x-7=\pm65$.
	6	$\therefore x=\frac{7\pm65}{18}=4$, or $-3\frac{2}{9}=-\frac{29}{9}$.

13. Given	1	$\frac{3x-4}{x-4}+\frac{x-2}{2}=9$.
(1) × $2(x-4)$	2	$6x-8+x^2-6x+8=18x-72$.
(2) Transposed	3	$x^2-18x=-72$.
Art. 82	4	$x^2-18x+81=9$.
Extracting root	5	$x-9=\pm3$.
	6	$\therefore x=9\pm3=12$, or 6.

SOLUTIONS OF QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. Let $x=$ the less, then $x+15$ will be the greater; and,

by the question,	1	$\frac{x(x+15)}{2}=x^3$.

$(1)\times\frac{2}{x}$	2	$x+15=2x^2$.
(2)transposed	3	$2x^2-x=15$.
Art. 81	4	$16x^2-8x+1=121$.
Extracting root	5	$4x-1=\pm 11$.
	6	$\therefore x=\frac{1\pm 11}{4}=3$, or $-2\frac{1}{2}$.

Hence $x+15=18$, or $12\frac{1}{2}$.

2. Let $x=$the less, then $100-x$ will be the greater; and,

by the question,	1	$x(100-x)=2059$.
	2	$100x-x^2=2059$.
changing signs of (2)	3	$x^2-100x=-2059$.
Art. 82.	4	$x^2-100x+2500=441$.
Extracting root	5	$x-50=\pm 21$
	6	$\therefore x=50\pm 21=71$ or 29.

3. Let $x=$the less, then $x+8$ will be the greater; and,

by the question,	1	$x(x+8)=240$.
	2	$x^2+8x=240$.
Art. 82	3	$x^2+8x+16=256$.
Extracting the root	4	$x+4=\pm 16$.
	5	$\therefore x=\pm 16-4=12$, or -20.
	6	$x+8=20$, or -12.

4. Let $x=$ the prime cost of the piece; then by the principles of "Loss and Gain," we have the following proportion:—

	1	$x:100::24:100+x$.
Art. 68	2	$x^2+100x=2400$.
Art. 82	3	$x^2+100x+2500=4900$.
Extracting root	4	$x+50=\pm 70$.
	5	$\therefore x=\pm 70-50=20$, or -120.

5. Let $x=$ the number of oxen bought; $\therefore \frac{480}{x}$ was the price in pounds paid for each; and since he retained six to himself, the number sold was $(x-6)$, and the price for which each was sold was $\frac{480}{x-6}$; but by the question,

	1	$\frac{480}{x}+4=\frac{480}{x-6}$.
$(1)\times x(x-6)$	2	$480x-2880+4x^2-24x=480x$.
(2)transposed	3	$4x^2-24x=2880$.
$(3)\div 4$	4	$x^2-6x=720$.
Art. 82	5	$x^2-6x+9=729$.
Extracting the root,	6	$x-3=\pm 27$.
	7	$x=3\pm 27=30$, or -24.

NOTE. In several of the preceding solutions, and in those that follow, it may be remarked, that of the two values found, only one is consistent with the conditions of the question, although both fulfil the conditions of the algebraic equation derived from it, thus showing that the algebraic equation is more general than the question from which it was derived. In the Algebra those values only are given which are consistent with the nature of the question.

6. Let $x=$ the length (and consequently the price in shillings of a yard) of the shortest trench; then $(x+4)$ will represent the length (and also the price in shillings) of a yard of the longest trench; ∴ by the question,

	1	$x^2+(x+4)^2=400=$ shillings in L.20.
collecting (1)	2	$2x^2+8x=384$.
(2)÷2	3	$x^2+4x=192$.
Art. 82	4	$x^2+4x+4=196$.
Extracting root	5	$x+2=\pm 14$.
	6	$\therefore x=\pm 14-2=12$, or -16.
	7	and $x+4=16$, or -12.

7. Let $x=$ the breadth of the frame, then the length of the glass and frame will be $(24+2x)$, and the width $(16+2x)$, and the surface of the glass and frame will be $(24+2x)(16+2x)$, while the surface of the glass is $24\times 16=384$; and by the question, the first is double of the second; whence

	1	$(24+2x)(16+2x)=768$.
	2	$4x^2+80x+384=768$.
(2)÷4, and transp.	3	$x^2+20x=96$.
Art. 82	4	$x^2+20x+100=196$.
Extracting root	5	$x+10=\pm 14$.
	6	$\therefore x=\pm 14-10=4$.

The other value is evidently inconsistent with the nature of the problem.

8. Let $x=$ the first; then since the sum of the first and second is 10, the second will be $(10-x)$; again, since they are in geometrical progression, $x:(10-x)::(10-x):\frac{(10-x)^2}{x}=$ the third; therefore by the question,

	1	$\frac{(10-x)^2}{x}-(10-x)=24$.
(1.) reduced, &c.	2	$2x^2-30x+100=24x$.
(2) transposed	3	$2x^2-54x=-100$.
Art. 81	4	$16x^2-432x+2916=2116$.
Extracting root	5	$4x-54=\pm 46$.
	6	$\therefore x=\frac{54\pm 46}{4}=25$ or 2.

The first value is inconsistent with the nature of the question, but the second is the number sought, and hence the numbers are 2, 8, and 32.

9. Let x=B's rate of travelling in miles per hour, then $(x+1)$ will be A's; and by the question,

	1	$\frac{300}{x+1}+10=\frac{300}{x}$.
$(1)\times x(x+1)$	2	$300x+10x^2+10x=300x+300$.
$(2)\div 10$ and transposed	3	$x^2+x=30$.
Art. 82	4	$x^2+x+\frac{1}{4}=\frac{121}{4}$.
	5	$x+\frac{1}{2}=\pm\frac{11}{2}$.
	6	$\therefore x=\pm\frac{11}{2}-\frac{1}{2}=5$, the value sought.

10. Let x=the persons relieved by B; then $(x+40)$ will be those relieved by A; also $\frac{1200}{x}$ = the pounds B gave to each, and $\frac{1200}{x+40}$ =the pounds A gave to each; but by the question the first is 5 greater than the second; therefore

	1	$\frac{1200}{x}-5=\frac{1200}{x+40}$
$(1)\times x(x+40)$	2	$1200x+48000-5x^2-200x=1200x$.
$(2)\div 5$ and transp.	3	$x^2+40x=9600$.
Art. 82	4	$x^2+40x+400=10000$.
Extracting root	5	$x+20=\pm 100$.
	6	$\therefore x=100-20=80$.
	7	and $x+40=80+40=120$.

11. Let x=the number of pieces; then $\frac{675}{x}$ is the number of shillings paid for a piece, and $675+\frac{675}{x}$ what the whole was sold for, also $=48x$; therefore

	1	$48x=675+\frac{675}{x}$.
$(1)\times x$ and transp.	2	$48x^2-675=675$.
$(2)\div 3$	3	$16x^2-225x=225$.
Formula, Art. 80	4	$x=\frac{225\pm\sqrt{(225)^2+14400}}{32}$ $=\frac{225\pm 255}{32}=15$ pieces.

12. Let *the number* who dined be x; then by the question,

	1	$\frac{75}{x}+2\frac{1}{2}=\frac{75}{x-1}$.
(1) × $2x(x-1)$	2	$150x-150+5x^2-5x=150x$.
(2) transposed	3	$5x^2-5x=150$.
(3) ÷ 5	4	$x^2-x=30$.
Art. 82	5	$x^2-x+\frac{1}{4}=\frac{121}{4}$.
	6	$x-\frac{1}{2}=\pm\frac{11}{2}$.
	7	$\therefore x=\frac{1}{2}\pm\frac{11}{2}=6$, the number sought.

13. The solution of this is the same as (Exercise 6), only put 6 instead of 4, and 68 shillings instead of 400.

14. Suppose the first had x eggs; then the second must have had $(100-x)$. Since therefore the former would have sold $(100-x)$ eggs for 15 pence, we have the following proportion: $(100-x):x::15:\frac{15x}{100-x}$, the sum the first received.

Also, since the second would have sold x eggs for $6\frac{2}{3}$ pence, and did sell $(100-x)$, we have the following proportion: $x:(100-x)::\frac{20}{3}:\frac{2000-20x}{3x}$, the sum the second received. But by the question the sum that each received was the same; hence

	1	$\frac{15x}{100-x}=\frac{2000-20x}{3x}$.
(1) × $3x(100-x)$	2	$45x^2=200000-4000x+20x^2$.
(2) transp.	3	$25x^2+4000x=200000$.
(3) ÷ 25	4	$x^2+160x=8000$.
Art. 82	5	$x^2+160x+6400=14400$.
Extracting root	6	$x+80=\pm120$.
	7	$\therefore x=\pm120-80=40$.

The other value is inconsistent with the conditions of the question.

Substituting this value of x in the expressions found above for the sum that each received, we find that each received 10 pence.

15. Since the numbers had equal differences, it is plain that the middle number will be a third part of their sum, and will therefore be 3. Let now $x=$ the difference between this and each of the others; then they will be represented by $(3-x)$ and $(3+x)$.

$$(3-x)^4=81-108x+54x^3-12x+x^4.$$
$$3^4=81.$$
$$(3+x)=81+108x+54x^2+12x+x^4.$$

Sum of fourth powers $=243+108x^2+2x^4=707$.
by the question:

hence per question,	1	$2x^4+108x^2+243=707$.
(1)÷2 and transposed	2	$x^4+54x^2=232$.
Art. 82	3	$x^4+54x^2+729=961$.
Extracting root	4	$x^2+27=31$.
	5	$\therefore x^2=31-27=4$.

and $x=2$; hence the numbers are $3-2=1$, 3, and $3+2=5$.

SOLUTION OF QUADRATICS WITH TWO UNKNOWN QUANTITIES.—Page 54.

1.	1	$x+y=12,$ the given equations.
	2	$xy=32,$
$(1)^2$	3	$x^2+2xy+y^2=144$.
$(2)\times 4$	4	$4xy=128$.
$(3)-(4)$	5	$x^2-2xy+y^2=16$.
Extracting root	6	$x-y=4$.
$\frac{1}{2}(1)+\frac{1}{2}(6)$	7	$x=8$, value of x.
$\frac{1}{2}(1)-\frac{1}{2}(6)$	8	$y=4$, value of y.

2.	1	$x-y=7$ The given equations.
	2	$xy=18$
$(1)^2$	3	$x^2-2xy+y^2=49$.
$(2)\times 4$	4	$4xy=72$.
$(3)+(4)$	5	$x^2+2xy+y^2=121$.
Extracting root	6	$x+y=11$.
$\frac{1}{2}(6)+\frac{1}{2}(1)$	7	$x=9$, value of x.
$\frac{1}{2}(6)-\frac{1}{2}(1)$	8	$y=2$, value of y.

3.	1	$x^2+y^2=169$ The given
	2	$x-y=7$ equations.
$(2)^2$	3	$x^2-2xy+y^2=49$.
$(1)-(3)$	4	$2xy=120$.
$(1)+(4)$	5	$x^2+2xy+y^2=289$.
Extracting root	6	$x+y=17$.
$\frac{1}{2}(6)+\frac{1}{2}(2)$	7	$x=12$, value of x.
$\frac{1}{2}(6)-\frac{1}{2}(2)$	8	$y=5$, value of y.

4.	1	$x^2-y^2=72$ The given
	2	$x^2+3y^2=108$ equations.
$(2)-(1)$	3	$4y^2=36$.
$\frac{1}{2}(3)^{\frac{1}{2}}$	4	$\therefore y=3$, value of y.
$(1)\times 3$	5	$3x^2-3y^2=216$.
$(5)+(2)$	6	$4x^2=324$.
$\frac{1}{2}(6)^{\frac{1}{2}}$	7	$x=9$, value of x.

5.	1	$x+2y=15$ The given equations.
	2	$x^2+4y^2=113$
$(1)^2$	3	$x^2+4xy+4y^2=225.$
$(3)-(2)$	4	$4xy=112.$
$(2)-(4)$	5	$x^2-4xy+4y^2=1.$
$(5)^{\frac{1}{2}}$	6	$x-2y=\pm 1.$
$\frac{1}{2}(1)+\frac{1}{2}(6)$	7	$x=8$ or 7, values of x.
$\frac{1}{4}(1)-\frac{1}{4}(6)$	8	$y=3\frac{1}{2}$ or 4, values of y.

6.	1	$x+4y=63$ The given equations.
	2	$\frac{4x}{y^2}=\frac{5}{12}$
(1) transposed	3	$x=63-4y.$
In (2) subst. (3)	4	$\frac{252-16y}{y^2}=\frac{5}{12}.$
(4) × $12y^2$ gives	5	$5y^2+192y=3024.$
Formula, Art. 80	6	$y=\frac{-192\pm\sqrt{(192)^2+20\times 3024}}{10}$
		$=\frac{-192\pm 312}{10}=12$, value of y.
Substituting in (1)	7	$x+48=63.$
	8	$\therefore x=15$, value of x.

7.	1	$xy=96$ The given equations
	2	$\frac{4x-y}{x}=2\frac{1}{2}$
(1) gives	3	$y=\frac{96}{x}.$
(2) gives	4	$y=1\frac{1}{2}x.$
	5	$1\frac{1}{2}x=\frac{96}{x}.$
$(5)\times\frac{2x}{3}$	6	$x^2=64.$
Extracting root	7	$x=8$, value of x.
(4)	8	$y=1\frac{1}{2}x=12$, value of y.

8.	1	$x^2-xy=20$ The given equations.
	2	$xy-y^2=16$
Let $y=vx$, (1)	3	$x^2-vx^2=20.$
and we have (2)	4	$vx^2-v^2x^2=16.$
$(3)\div(4)$ gives	5	$\frac{1}{v}=\frac{5}{4}.$
$(5)\times\frac{4v}{5}$	6	$\frac{4}{5}=v.$
Subst. for v in (3)	7	$x^2-\frac{4}{5}x^2=20.$
	8	$\therefore x=10$, value of x.

Subst. for x in (1)	9	$100-10y=20.$
	10	$\therefore y=8$, value of y.

9. Let $x =$ the greater, and $y =$ the less; then by the

question,	1	$x^2+xy=240.$
	2	$xy+y^2=160.$
(1)+(2)	3	$x^2+2xy+y^2=400.$
Extracting root	4	$x+y=20.$
(1)÷(4) gives	5	$x=12$, the greater.
(2)÷(4) gives	6	$y=8$, the less.

10. Let the notation be the same as in the last; then by

the question,	1	$x^2+xy=220.$
	2	$xy-y^2=18.$
Let $y=vx$, and we have $\begin{cases}(1)\\(2)\end{cases}$	3	$x^2(1+v)=220.$
	4	$x^2(v-v^2)=18.$
(3)÷(4) gives	5	$\frac{1+v}{v-v^2}=\frac{110}{9}.$
(5)×9$(v-v^2)$	6	$9+9v=110v-110v^2$
(6) transposed	7	$110v^2-101v=-9.$
Formula, Art. 80	8	$v=\frac{101\pm\sqrt{(101)^2-440\times9}}{220}$
		$=\frac{101\pm79}{220}=\frac{9}{11}$ or $-\frac{1}{10}$.

Substitute these values of v successively in equation (4), and they give $\frac{18}{121}x^2=18$, or $x=11$; and $\frac{9}{100}x^2=18$, or $x=10\sqrt{2}$.

These values of x, multiplied by the corresponding values of v from which they were derived, give $y=9$, or $\sqrt{2}$.

11. Let the three numbers be represented by x, y, and z; then by the question,

	1	$x(x+y+z)=48.$
	2	$y(x+y+z)=96.$
	3	$z(x+y+z)=112.$
(1)+(2)+(3)	4	$(x+y+z)^2=256.$
Extracting root	5	$x+y+z=16.$
(1)÷(5) gives	6	$x=3$, value of the first.
(2)÷(5) gives	7	$y=6$, value of the second.
(3)÷(5) gives	8	$z=7$, value of the third.

12. Let $x =$ the greater, and $y =$ the less; then by the

question,	1	$5x-5y=4y.$
	2	$x^2+4y^2=181.$
(1) gives	3	$y=\frac{5}{9}x.$
Substituting in (2)	4	$x^2+\frac{100}{81}x^2=181.$

(4)×81 gives	5	$181x^2=181\times81.$
(5)÷81 gives	6	$x^2=81.$
	7	$\therefore x=9$, value of the greater.
Substituting in (3)	8	$y=\frac{5}{9}9=5$, value of the less.

13. Let the greater be represented by $(x+y)$, and the less by $(x-y)$; then by the question,

	1	$(x+y)^2+(x-y)^2=34.$
	2	$(x+y)^4+(x-y)^4=706.$
(1) expanded, &c.	3	$x^2+y^2=17.$
(2) ditto	4	$x^4+6x^2y^2+y^4=353.$
(3) squared	5	$x^4+2x^2y^2+y^4=289.$
(4)—(5) gives	6	$4x^2y^2=64.$
Extracting root	7	$2xy=8.$
(3)+(7)	8	$x^2+2xy+y^2=25.$
Extracting root	9	$x+y=5$, the greater number.
(3)—(7)	10	$x^2-2xy+y^2=9.$
Extracting root	11	$x-y=3$, the less number.

14. Let $x =$ the numerator, and y the denominator.

By the question,	1	$\frac{x}{y}=2\frac{x^2}{y^2}.$
	2	$(x+1)x=(y-1)y.$
(1)×$\frac{y^2}{x}$ gives	3	$y=2x.$
Subst. for y in (2)	4	$x^2+x=4x^2-2x.$
(4)÷x and transp.	5	$3x=3.$
	6	$\therefore x=1$, value of the numerator.
Subst. in (2) gives	7	$y^2-y=2.$
Art. 82	8	$y^2-y+\frac{1}{4}=\frac{9}{4}.$
Extracting root	9	$y-\frac{1}{2}=\pm\frac{3}{2}.$
	10	$\therefore y=\frac{1}{2}\pm\frac{3}{2}=2$, or -1, denominator.

The last value is not consistent with the nature of the question. Hence the fraction sought is $\frac{1}{2}$.

SURDS.

1. $ax=(ax\times ax)^{\frac{1}{2}}=(a^2x^2)^{\frac{1}{2}},$

$$3ay=(3ay\times 3ay)^{\frac{1}{2}}=(9a^2y^2)^{\frac{1}{2}},$$

$$\frac{2a}{c}=\left(\frac{2a}{c}\times\frac{2a}{c}\right)^{\frac{1}{2}}=\left(\frac{4a^2}{c^2}\right)^{\frac{1}{2}},$$

$$\frac{4a^2}{3x^3}=\left(\frac{4a^2}{3x^3}\times\frac{4a^2}{3x^3}\right)^{\frac{1}{2}}=\left(\frac{16a^4}{9x^6}\right)^{\frac{1}{2}},$$

$\frac{5a}{4c^2}=\left(\frac{5a}{4c^2}\times\frac{5a}{4c^2}\right)^{\frac{1}{2}}=\left(\frac{25a^2}{16c^4}\right)^{\frac{1}{2}}$,

$\frac{3}{4}y^2=(\frac{3}{4}y^2\times\frac{3}{4}y^2)^{\frac{1}{2}}=\left(\frac{9}{16}y^4\right)^{\frac{1}{2}}=\sqrt{\frac{9y^4}{16}}$.

2. $2a^2c=(2a^2c\times 2a^2c\times 2a^2c)^{\frac{1}{3}}=(8a^6c^3)^{\frac{1}{3}}$,

$\frac{3a}{c^2}=\left(\frac{3a}{c^2}\times\frac{3a}{c^2}\times\frac{3a}{c^2}\right)^{\frac{1}{3}}=\left(\frac{27a^3}{c^6}\right)^{\frac{1}{3}}$,

$4ay^2=(4ay^2\times 4ay^2\times 4ay^2)^{\frac{1}{3}}=(64a^3y^6)^{\frac{1}{3}}$,

$3a^nx=(3a^nx\times 3a^nx\times 3a^nx)^{\frac{1}{3}}=(27a^{3n}x^3)^{\frac{1}{3}}$,

$\frac{4ac}{2x}=\left(\frac{4ac}{2x}\times\frac{4ac}{2x}\times\frac{4ac}{2x}\right)^{\frac{1}{3}}=\left(\frac{64a^3c^3}{8x^3}\right)^{\frac{1}{3}}$.

3. $-2a=(-2a\times-2a\times-2a\times-2a\times-2a)^{\frac{1}{5}}=(-32a^5)^{\frac{1}{5}}$,

$\frac{3cx}{a}=\left(\frac{3cx}{a}\times\frac{3cx}{a}\times\frac{3cx}{a}\times\frac{3cx}{a}\times\frac{3cx}{a}\right)^{\frac{1}{5}}=\left(\frac{243c^5x^5}{a^5}\right)^{\frac{1}{5}}$,

$\frac{ax}{y^2}=\left(\frac{ax}{y^2}\times\frac{ax}{y^2}\times\frac{ax}{y^2}\times\frac{ax}{y^2}\times\frac{ax}{y^2}\right)^{\frac{1}{5}}=\left(\frac{a^5x^5}{y^{10}}\right)^{\frac{1}{5}}$,

$-\frac{x^2y}{mn^2}=\left((-)^5\times\frac{(x^2)^5\times y^5}{m^5\times(n^2)^5}\right)^{\frac{1}{5}}=\left(-\frac{x^{10}y^5}{m^5n^{10}}\right)^{\frac{1}{5}}$,

$\frac{3a^n}{y^2}=\left(\frac{(3)^5\times(a^n)^5}{(y^2)^5}\right)^{\frac{1}{5}}=\left(\frac{243a^{5n}}{y^{10}}\right)^{\frac{1}{5}}$.

4. $4\sqrt{a}=\sqrt{4\times 4}\times\sqrt{a}=\sqrt{16a}$,

$3\sqrt[3]{ax^2}=\sqrt[3]{3\times 3\times 3}\times\sqrt[3]{ax^2}=\sqrt[3]{27ax^2}=(27ax^2)^{\frac{1}{3}}$,

$5(ax^2)^{\frac{1}{4}}=(5\times 5\times 5\times 5)^{\frac{1}{4}}(ax^2)^{\frac{1}{4}}=(625ax^2)^{\frac{1}{4}}$,

$3(x^2y^3)^{\frac{1}{4}}=(3\times 3\times 3\times 3)^{\frac{1}{4}}(x^2y^3)^{\frac{1}{4}}=(81x^2y^3)^{\frac{1}{4}}$.

(86.)

1. $\sqrt{32a^5}=\sqrt{16a^4\times 2a}=\sqrt{16a^4}\times\sqrt{2a}=4a^2\sqrt{2a}$.

$\sqrt[3]{81a^7}=\sqrt[3]{27a^6\times 3a}=\sqrt[3]{27a^6}\times\sqrt[3]{3a}=3a^2\sqrt[3]{3a}$.

$\sqrt{125}=\sqrt{25\times 5}=\sqrt{25}\times\sqrt{5}=5\sqrt{5}$.

$(180a^3x^2)^{\frac{1}{2}}=(36a^2x^2\times 5a)^{\frac{1}{2}}=(36a^2x^2)^{\frac{1}{2}}\times(5a)^{\frac{1}{2}}=6ax(5a)^{\frac{1}{2}}$.

2. $\sqrt[3]{1250x^4y^5}=\sqrt[3]{125x^3y^3\times 10xy^2}=5xy\sqrt[3]{10xy^2}$.

$(96a^5x^2)^{\frac{1}{4}}=(16a^4\times 6ax^2)^{\frac{1}{4}}=2a(6ax^2)^{\frac{1}{4}}$.

$(72x^5y^3)^{\frac{1}{2}}=(36x^4y^2\times 2xy)^{\frac{1}{2}}=6x^2y(2xy)^{\frac{1}{2}}$.

3. $\sqrt{\frac{a^3x^2}{7}}=\sqrt{\frac{a^2x^2}{49}\times 7a}=\frac{ax}{7}\sqrt{7a}.$

$\sqrt[3]{\frac{a^4c^3x^5}{9}}=\sqrt[3]{\frac{a^3c^3x^3}{27}\times 3ax^2}=\frac{acx}{3}\sqrt[3]{3ax^2}.$

$\left(\frac{4a^5x^6}{27}\right)^{\frac{1}{2}}=\left(\frac{4a^4x^6}{81}\times 3a\right)^{\frac{1}{2}}=\frac{2a^2x^3}{9}\sqrt{3a}.$

4. $\left(\frac{3x^2z}{4}\right)^{\frac{1}{3}}=\left\{\frac{1}{8}\times(6x^2z)\right\}^{\frac{1}{3}}=\frac{1}{2}(6x^2z)^{\frac{1}{3}}.$

$\left(\frac{5a^4x^3}{36}\right)^{\frac{1}{3}}=\left(\frac{a^3x^3}{216}\times 30a\right)^{\frac{1}{3}}=\frac{ax}{6}(30a)^{\frac{1}{3}}.$

$\left(\frac{7a^3y^5}{3}\right)^{\frac{1}{2}}=\left(\frac{a^2y^4}{9}\times 21ay\right)^{\frac{1}{2}}=\frac{ay^2}{3}(21ay)^{\frac{1}{2}}.$

$\left(\frac{5xy^2z^5}{8}\right)^{\frac{1}{4}}=\left(\frac{z^4}{16}\times 10xy^2z\right)^{\frac{1}{4}}=\frac{z}{2}(10xy^2z)^{\frac{1}{4}}.$

(87.) 1. Since $\frac{1}{3}$ and $\frac{1}{2}$ are $=\frac{2}{6}$ and $\frac{3}{6}$, $\therefore (ac)^{\frac{1}{3}}$ and $(5)^{\frac{1}{2}}$ $=(ac)^{\frac{2}{6}}$ and $(5)^{\frac{3}{6}}=(a^2c^2)^{\frac{1}{6}}$ and $(125)^{\frac{1}{6}}$.

2. Since $\frac{1}{5}$ and $\frac{1}{4}=\frac{4}{20}$ and $\frac{5}{20}$, $\therefore 4^{\frac{1}{5}}$ and $3^{\frac{1}{4}}=4^{\frac{4}{20}}$ and $3^{\frac{5}{20}}=(256)^{\frac{1}{20}}$ and $(243)^{\frac{1}{20}}$.

3. $\sqrt[3]{4a^2x}=(4a^2x)^{\frac{1}{3}}$ and $\sqrt[4]{3a^3x^2}=(3a^3x^2)^{\frac{1}{4}}$. But $\frac{1}{3}=\frac{4}{12}$ and $\frac{1}{4}=\frac{5}{12}$, $\therefore (4a^2x)^{\frac{1}{3}}$ and $(3a^3x^2)^{\frac{1}{4}}=(4a^2x)^{\frac{4}{12}}$ and $(3a^3x^2)^{\frac{5}{12}}=(256a^8x^4)^{\frac{1}{12}}$ and $(27a^9x^6)^{\frac{1}{12}}$.

4. Since $\frac{1}{3}$ and $\frac{1}{5}=\frac{5}{15}$ and $\frac{3}{15}$, $\therefore \left(\frac{ac}{3}\right)^{\frac{1}{3}}$ and $\left(\frac{4}{ac}\right)^{\frac{1}{5}}=$ $\left(\frac{ac}{3}\right)^{\frac{5}{15}}$ and $\left(\frac{4}{ac}\right)^{\frac{3}{15}}=\left(\frac{a^5c^5}{243}\right)^{\frac{1}{15}}$ and $\left(\frac{64}{a^3c^3}\right)^{\frac{1}{15}}$.

5. Since $\frac{1}{2}=\frac{2}{4}$, the first can be reduced to the same fractional exponent as the second. $\left(\frac{ab}{2c}\right)^{\frac{1}{2}}=\left(\frac{ab}{2c}\right)^{\frac{2}{4}}=\left(\frac{a^2b^2}{4c^2}\right)^{\frac{1}{4}}$.

(88.) To Add and Subtract Surds.

1. $3\sqrt{32}=3\sqrt{16\times 2}=12\sqrt{2}$, and $2\sqrt{162}=$ $2\sqrt{81\times 2}=18\sqrt{2}$, $\therefore 18\sqrt{2}+12\sqrt{2}=30\sqrt{2}$, sum,
and $18\sqrt{2}-12\sqrt{2}=6\sqrt{2}$, diff.

2. $3\sqrt[3]{54}=3\sqrt[3]{27\times 2}=9\sqrt[3]{2}$ and $\sqrt[3]{250}=\sqrt[3]{125\times 2}$ $=5\sqrt[3]{2}$ $\therefore 9\sqrt[3]{2}+5\sqrt[3]{2}=14\sqrt[3]{2}$, sum,
and $9\sqrt[3]{2}-5\sqrt[3]{2}=4\sqrt[3]{2}$, difference.

3. $\sqrt[3]{24a^4}=\sqrt[3]{8a^3\times 3a}=2a\sqrt[3]{3a}$ and $\sqrt[3]{192a}=$ $\sqrt[3]{64\times 3a}=4\sqrt[3]{3a}$, $\therefore 2a\sqrt[3]{3a}+4\sqrt[3]{3a}=(2a+4)\sqrt[3]{3a}$, sum,

and $2a\sqrt[3]{3a}-4\sqrt[3]{3a}=(2a-4)\sqrt[3]{3a}$, difference.

4. $\sqrt{80}=\sqrt{16\times 5}=4\sqrt{5}$ and $\sqrt{45}=\sqrt{9\times 5}=3\sqrt{5}$

$\therefore 4\sqrt{5}+3\sqrt{5}=7\sqrt{5}$, sum,

and $4\sqrt{5}-3\sqrt{5}=\sqrt{5}$, difference.

5. $(36a^3)^{\frac{1}{2}}=(36a^2\times a)^{\frac{1}{2}}=6aa^{\frac{1}{2}}$, and $(98a)^{\frac{1}{2}}=(49\times 2a)^{\frac{1}{2}}$ $=7(2a)^{\frac{1}{2}}$;

$\therefore 6a\sqrt{a}+7\sqrt{2a}=$ their sum,

and $6a\sqrt{a}-7\sqrt{2a}=$ their difference.

6. $(1000a^5)^{\frac{1}{2}}=\sqrt{100a^4\times 10a}=10a^2\sqrt{10a}$, and

$(300a^3)^{\frac{1}{2}}=\sqrt{100a^2\times 3a}=10a\sqrt{3a}$.

$\therefore 10a^2\sqrt{10a}+10a\sqrt{3a}$, =their sum,

and $10a^2\sqrt{10a}-10a\sqrt{3a}$, =their difference.

(89.) To Multiply Surds.

1. $5\sqrt{5}\times 3\sqrt{8}=15\sqrt{40}=15\sqrt{4\times 10}=30\sqrt{10}$.

2. $\sqrt[3]{18}\times 5\sqrt[3]{4}=5\sqrt[3]{72}=5\sqrt[3]{8\times 9}=10\sqrt[3]{9}$.

3. $\sqrt{10}=(10)^{\frac{3}{6}}=(1000)^{\frac{1}{6}}$, and $\sqrt[3]{15}=(15)^{\frac{2}{6}}=(225)^{\frac{1}{6}}$.

$\therefore (1000)^{\frac{1}{6}}\times(225)^{\frac{1}{6}}=\sqrt[6]{225000}$.

or $\sqrt{10}=(2\times 5)^{\frac{3}{6}}=(2^3.5^3)^{\frac{1}{6}}$, and $\sqrt[3]{15}=(3\times 5)^{\frac{2}{6}}=$ $(3^2.5^2)^{\frac{1}{6}}$.

$\therefore (2^3.5^3)^{\frac{1}{6}}\times(3^2.5^2)^{\frac{1}{6}}=(2^3.3^2.5^5)^{\frac{1}{6}}$.

4. Since $\frac{2}{3}=\frac{8}{12}$ and $\frac{1}{4}=\frac{3}{12}$; $\therefore 5a^{\frac{2}{3}}\times\frac{1}{2}a^{\frac{1}{4}}=5a^{\frac{8}{12}}\times\frac{1}{2}a^{\frac{3}{12}}$ $=\frac{5}{2}a^{\frac{8}{12}+\frac{3}{12}}=\frac{5}{2}a^{\frac{11}{12}}$.

5. $(a+b)^{\frac{1}{2}}\times(a-b)^{\frac{1}{2}}=\{(a+b)(a-b)\}^{\frac{1}{2}}=(a^2-b^2)^{\frac{1}{2}}$.

6. Since $\frac{m}{n}=\frac{ms}{ns}$, and $\frac{r}{s}=\frac{nr}{ns}$; $\therefore a^{\frac{m}{n}}\times a^{\frac{r}{s}}=a^{\frac{ms}{ns}}\times$ $a^{\frac{nr}{ns}}=a^{\frac{ms}{ns}+\frac{nr}{ns}}=a^{\frac{ms+nr}{ns}}$.

(90.) TO DIVIDE SURDS.

1. $\frac{10\sqrt{27}}{2\sqrt{3}}=5\sqrt{\frac{27}{3}}=5\sqrt{9}=5\times3=15.$

2. $\frac{\frac{1}{2}\sqrt[3]{6}}{\frac{2}{3}\sqrt[3]{6}}=\frac{1}{2}\times\frac{3}{2}\sqrt[3]{\frac{6}{6}}=\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}.$

3. $\frac{4\sqrt[3]{18}}{2\sqrt[3]{9}}=2\sqrt[3]{\frac{18}{9}}=2\sqrt[3]{2}.$

4. $\frac{\frac{1}{2}\sqrt{a}}{\frac{2}{3}\sqrt[3]{ab}}=\frac{1}{2}\times\frac{3}{2}\frac{a^{\frac{3}{6}}}{(ab)^{\frac{2}{6}}}=\frac{3}{4}\left(\frac{a^3}{a^2b^2}\right)^{\frac{1}{6}}=\frac{3}{4}\left(\frac{a}{b^2}\right)^{\frac{1}{6}}=\frac{3}{4}\left(\frac{ab^4}{b^6}\right)^{\frac{1}{6}}$

$=\frac{3}{4}\left(\frac{1}{b^6}\times ab^4\right)^{\frac{1}{6}}=\frac{3}{4b}(ab^4)^{\frac{1}{6}}.$

5. $\frac{\frac{3}{4}\sqrt{a^3}}{\frac{3}{5}\sqrt[3]{a^2}}=\frac{3}{4}\times\frac{5}{3}\frac{(a^3)^{\frac{1}{2}}}{(a^2)^{\frac{1}{3}}}=\frac{5}{4}\frac{(a^9)^{\frac{1}{6}}}{(a^4)^{\frac{1}{6}}}=\frac{4}{5}\left(\frac{a^9}{a^4}\right)^{\frac{1}{6}}=\frac{4}{5}a^{\frac{5}{6}}.$

6. $\frac{(a^2-b^2)^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}}=\left(\frac{(a+b)(a-b)}{a-b}\right)^{\frac{1}{2}}=(a+b)^{\frac{1}{2}}.$

(91.) INVOLUTION OF SURDS.

1. The second power of 2 is 4, and the second power of $(ac)^{\frac{1}{3}}$ is $(ac)^{\frac{1}{3}\times2}=(ac)^{\frac{2}{3}}$; $\therefore$ the quantity sought is $4(ac)^{\frac{2}{3}}$.

2. 4 raised to the third power is $4\times4\times4=64$, and $(bcx^2)^{\frac{1}{2}}$,......$(bcx^2)^{\frac{1}{2}\times3}=(bcx^2)^{\frac{3}{2}}=bcx^2(bcx^2)^{\frac{1}{2}}=bcx^3(bc)^{\frac{1}{2}}$; $\therefore$ the quantity sought is $64bcx^3(bc)^{\frac{1}{2}}$.

3. The fourth power is the square of the square, and the square of $\frac{1}{6}\sqrt{6}=\frac{6}{36}=\frac{1}{6}$, which being again squared, gives $\frac{1}{36}$. Or $\frac{1}{6}\sqrt{6}=\frac{6^{\frac{1}{2}}}{6}=\frac{1}{6^{\frac{1}{2}}}$, which raised to the fourth power, $=\frac{1}{(6^{\frac{1}{2}})^4}=\frac{1}{6^2}=\frac{1}{36}$.

4. Here the exponent of a multiplied by 6 is 3, and the exponent of b multiplied by 6 is 2. Hence the quantity sought is a^3b^2.

5. Applying (Art. 57), we have $1^3=1$ for the first term; $3\times1^2\times\sqrt{x}=3\sqrt{x}$ for the second; $3\sqrt{x}\times\frac{2}{2}\times\frac{\sqrt{x}}{1}=3x$

for the third; and $3x \times \frac{1}{3} \times \frac{\sqrt{x}}{1} = x\sqrt{x}$ for the fourth. And continuing the same process, we will find the next term and all after become 0; $\therefore$ the power sought will evidently be $1+3\sqrt{x}+3x+x\sqrt{x}$.

6. By (Art. 29), $(3+2\sqrt{5})^2=3^2+(2\sqrt{5})^2+2\times3\times2\sqrt{5}=29+12\sqrt{5}$.

(92.) EVOLUTION OF SURDS.

1. The square root of 9 is 3, and the square root of $3^{\frac{1}{3}}$ is $=3^{\frac{1}{3}\times\frac{1}{2}}=3^{\frac{1}{6}}$, or $\sqrt[6]{3}$; hence the quantity sought is $3\sqrt[6]{3}$.

2. The square root of 36 is 6, and the square root of $\sqrt[3]{2}=2^{\frac{1}{3}\times\frac{1}{2}}=2^{\frac{1}{6}}$, or $\sqrt[6]{2}$; hence the quantity sought is $6\sqrt[6]{2}$.

3. The cube root of 8 is 2, and the cube root of $\sqrt[3]{5}$ $=5^{\frac{1}{3}\times\frac{1}{3}}=5^{\frac{1}{9}}=\sqrt[9]{5}$; hence the quantity sought is $2\sqrt[9]{5}$.

4. The cube root of 27 is 3, and the cube root of $\sqrt{7}$ is $7^{\frac{1}{2}\times\frac{1}{3}}=7^{\frac{1}{6}}=\sqrt[6]{7}$; hence the quantity sought is $3\times7^{\frac{1}{6}}$, or $3\sqrt[6]{7}$.

5. The fourth root of 64 is $(64)^{\frac{1}{4}}=(64)^{\frac{3}{12}}=(2^{18})^{\frac{1}{12}}=2(2^6)^{\frac{1}{12}}$, and the fourth root of $\sqrt[3]{4}$ is $4^{\frac{1}{3}\times\frac{1}{4}}=4^{\frac{1}{12}}=(2^2)^{\frac{1}{12}}$; hence the quantity sought is $2(2^6)^{\frac{1}{12}}(2^2)^{\frac{1}{12}}=2\,(2^8)^{\frac{1}{12}}=2(256)^{\frac{1}{12}}$.

(94.) EXERCISES IN EQUATIONS.

1. $\sqrt{3x+4}=5$. Given equation.
$3x+4=25$, by squaring both sides.
$\therefore x=7$, by transp., and dividing by 3.

2. $\sqrt{4+5x}=2+\sqrt{3x}$. Given equation.
$4+5x=4+4\sqrt{3x}+3x$, by squaring both sides.
$2x=4\sqrt{3x}$, by transposition.
$4x^2=48x$, by squaring both sides.
$\therefore x=12$, by dividing by $4x$.

3. $\sqrt[3]{2x+10}+4=8$, given equation,
$\sqrt[3]{2x+10}=4$, by transposition,
$2x+10=64$, by cubing both sides,
$\therefore x=27$, by transposition and division by 2.

4. $\sqrt{\frac{2x}{3}}+5=7$, given equation,
$\frac{2x}{3}=4$, transposing and squaring,
$\therefore x=6$, by dividing by $\frac{2}{3}$.

5. $\sqrt{4x+17}+6\sqrt{x}+2=8\sqrt{x}+3$, given equation,
$\sqrt{4x+17}=2\sqrt{x}+1$, by transposition,
$4x+17=4x+4\sqrt{x}+1$, by squaring,
$\sqrt{x}=4$, by transposing and dividing by 4,
$\therefore x=16$, by squaring.

6. $\sqrt{x}+\sqrt{x-7}=\frac{21}{\sqrt{x-7}}$, given equation,
$\sqrt{x^2-7x}+x-7=21$, by mult. by $\sqrt{x-7}$,
$\sqrt{x^2-7x}=28-x$, by transposition,
$x^2-7x=784-56x+x^2$, by squaring,
$49x=784$, by transposition,
$\therefore x=16$, by dividing by 49.

7. $\frac{8x+4}{\sqrt{5+x}}=4\sqrt{5+x}$, given equation,
$8x+4=20+4x$, by multiplying by $\sqrt{5+x}$,
$\therefore x=4$, by transposing and dividing by 4.

8. $\sqrt{7+x}+\frac{1}{\sqrt{x}}=\frac{13}{\sqrt{x}}$, given equation,
$\sqrt{7x+x^2}+1=13$, by multiplying by $\sqrt{x}$,
$x^2+7x=144$, by transposing and squaring,
$x^2+7x+\frac{49}{4}=\frac{625}{4}$, completing the square,
$x+\frac{7}{2}=\pm\frac{25}{2}$, by extracting the root,
$\therefore x=9$ or -16.

9. $\frac{1}{\sqrt{1+x}}+\frac{2}{3}=\frac{(1+x)^{\frac{1}{2}}}{3}$, given equation,
$3+2\sqrt{1+x}=1+x$, by multiplying by $3\sqrt{1+x}$,
$4+4x=x^2-4x+4$, transposing and squaring,
$8x=x^2$, by transposing,
$\therefore x=8$.

10. $\sqrt{x}+\sqrt{x+a}=\frac{2a}{\sqrt{x+a}}$, given equation,

$\sqrt{x^2+ax}+x+a=2a$, by multiplying by $\sqrt{x+a}$,

$x^2+ax=a^2-2ax+x^2$, transposing and squaring,

$3ax=a^2$, by transposition,

$\therefore x=\frac{a}{3}$, by dividing by $3a$.

11. $\sqrt{x}-\sqrt{a}=\sqrt{ax}$, given equation,

$(1-\sqrt{a})\sqrt{x}=\sqrt{a}$, by transposition,

$\sqrt{x}=\frac{\sqrt{a}}{1-\sqrt{a}}$, by dividing by $1-\sqrt{a}$.

$\therefore x=\frac{a}{(1-\sqrt{a})^2}$, by squaring.

12. $\sqrt{x+a}+\sqrt{a-x}=b$, given equation,

$x+a+2\sqrt{a^2-x^2}+a-x=b^2$, squaring both sides,

$2\sqrt{a^2-x^2}=b^2-2a$, by transposition,

$4a^2-4x^2=b^4-4ab^2+4a^2$, by squaring,

$4x^2=b^2(4a-b^2)$, transp. & changing signs,

$\therefore x=\frac{b}{2}(4a-b^2)^{\frac{1}{2}}$, div. by 4 and ext. root.

13. $\sqrt{1+x\sqrt{x^2+12}}=1+x$, given equation,

$1+x\sqrt{x^2+12}=1+2x+x^2$, squaring both sides,

$\sqrt{x^2+12}=2+x$, subt. 1 and dividing by x,

$x^2+12=4+4x+x^2$, by squaring,

$\therefore x=2$, transposing and dividing by 4.

14. $\sqrt{x}+\sqrt{16+x}=\frac{40}{\sqrt{16+x}}$, given equation,

$\sqrt{16x+x^2}+16+x=40$, by mult. by $\sqrt{16+x}$,

$\sqrt{16x+x^2}=24-x$, by transposition,

$16x+x^2=576-48x+x^2$, by squaring,

$64x=576$, by transposition,

$\therefore x=9$.

15. $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$, given equation,

$\frac{2\sqrt{x}+32}{24}=\frac{2\sqrt{x}+44}{32}$, see (Art. 69).

$64\sqrt{x}+1024=48\sqrt{x}+1056$, by mult. both sides by 768,

$16\sqrt{x}=32$, by transposition,

$\therefore x=4$, by dividing by 16 and squaring.

16. $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\frac{a}{b}$, given equation,

$\frac{2\sqrt{x}}{2\sqrt{b}}=\frac{a+b}{a-b}$, (Art. 69),

$\frac{x}{b}=\frac{(a+b)^2}{(a-b)^2}$, by squaring and reducing fraction,

$\therefore x=b\left(\frac{a+b}{a-b}\right)^2$.

17. $\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$, given equation,

$\frac{\cancel{2}\sqrt{6x}}{2\,\cancel{4}}=\frac{8\sqrt{6x}-3}{15}$, (Art. 69),

$15\sqrt{6x}=16\sqrt{6x}-6$, clearing of fractions,

$\sqrt{6x}=6$, transposing and changing the signs,

$\therefore x=6$, by squaring and dividing by 6.

18. $x+2\sqrt{x}=24$, given equation,

$x+2\sqrt{x}+1=25$, completing the square,

$\sqrt{x}+1=\pm5$, by extracting the root,

$\therefore x=16$ or 36, by transposing and squaring.

19. $x^{\frac{5}{2}}-2x^{\frac{1}{2}}=x$, given equation,

$x-2=x^{\frac{1}{2}}$, by dividing by $x^{\frac{1}{2}}$,

$x-x^{\frac{1}{2}}=2$, by transposition,

$x-x^{\frac{1}{2}}+\frac{1}{4}=\frac{9}{4}$, by completing the square,

$x^{\frac{1}{2}}-\frac{1}{2}=\pm\frac{3}{2}$, extracting the root,

$x^{\frac{1}{2}}=2$ or -1, by transposition,

$\therefore x=4$ or 1, by squaring.

20. $x^{\frac{2}{5}}+x^{\frac{1}{5}}=6$, given equation,

$x^{\frac{2}{5}}+x^{\frac{1}{5}}+\frac{1}{4}=\frac{25}{4}$, (Art. 94), by completing square,

$x^{\frac{1}{5}}+\frac{1}{2}=\pm\frac{5}{2}$, by extracting the root,

$x^{\frac{1}{5}}=2$ or -3, by transposition,

$\therefore x=32$ or -243, by raising to the fifth power.

21. $3x-2x^{\frac{1}{2}}=133$, given equation,

Solving for $x^{\frac{1}{2}}$ by formula, (Art. 80), we have

$$x^{\frac{1}{2}}=\frac{2\pm40}{6}=7 \text{ or } -\tfrac{19}{3},$$

$\therefore x=49$ or $\frac{361}{9}$, by squaring both sides.

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22. $(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}=6$, given equation.

This being of the form of an adfected quadratic, in which the unknown quantity is $(x+12)^{\frac{1}{4}}$, we must first solve for $(x+12)^{\frac{1}{4}}$, and then raising both sides to the fourth power, solve the result again for x.

$(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}+\frac{1}{4}=\frac{25}{4}$, completing the square,
$(x+12)^{\frac{1}{4}}+\frac{1}{2}=\pm\frac{5}{2}$, extracting the root,
$(x+12)^{\frac{1}{4}}=2$ or -3, by transposition,
$x+12=16$ or 81, raising to the fourth power,
$\therefore x=4$ or 69.

(95.) ARITHMETICAL PROGRESSION.

Proof of Theorems, page 64.

$$\left.\begin{array}{c|l} 1 & l=a+\overline{n-1}d \\ 2 & s=\left\{2a+\overline{n-1}d\right\}\frac{n}{2} \end{array}\right\}\text{Given theorems.}$$

1. Since $l=a+\overline{n-1}d$, by equation (1),
$l-\overline{n-1}d=a$, by transposing $\overline{n-1}d$,
$\therefore a=l-\overline{n-1}d$. Q. E. D.

---

2. Since $s=\left\{2a+\overline{n-1}d\right\}\frac{n}{2}$, by equation (2.)

$2s=2an+n\times\overline{n-1}d$.
$2an=2s-n\times\overline{n-1}d$, by transposition.
$\therefore a=\frac{s}{n}-\frac{n-1}{2}d$. Q. E. D.

---

3. From (1) $\overline{n-1}d=l-a$, and

from (2) $\overline{n-1}d=\frac{2s}{n}-2a$. Equating these values,

we obtain $l-a=\frac{2s}{n}-2a$;

$\therefore a=\frac{2s}{n}-l$, by transp. Q. E. D.

4. Since $l=a+\overline{n-1}d$, (1); $\therefore n-1=\frac{l-a}{d}$.

Hence $n=\frac{l-a}{d}+1=\frac{l-a+d}{d}$.

Substitute these values of $\overline{n-1}$ and n in (2), and we obtain $s=\left\{2a+\frac{l-a}{d}d\right\}\frac{l-a+d}{2d}$;

$$\text{hence } 2ds=(2a+l-a)(l-a+d),\text{ mult. by } 2d,$$

$$\left.\begin{aligned}&=(l+a)(l-a+d);\\ \therefore\ 2ds&=l^2-a^2+ld+ad.\end{aligned}\right\}\text{ Reducing,}$$

$$a^2-da=l^2+ld-2ds,\text{ by transposition,}$$

$$a^2-da+\frac{d^2}{4}=l^2+ld+\frac{d^2}{4}-2ds\text{ compt. square,}$$

$$=\tfrac{1}{4}\{(2l+d)^2-8ds\};$$

$$\text{hence } a-\frac{d}{2}=\pm\tfrac{1}{2}\sqrt{(2l+d)^2-8ds}.\quad\text{Ext. root.}$$

$$\therefore a=\pm\tfrac{1}{2}\sqrt{(2l+d)^2-8ds}+\tfrac{1}{2}d.\qquad\text{Q. E. D.}$$

5. This is one of the given equations, from which the others are to be derived, viz. (1).

6. The value of $\overline{n-1}d$ determined from (2) is $\frac{2s-2an}{n}$; this being substituted instead of it in the first, gives

$$l=a+\frac{2s-2an}{n},$$

$$=a+\frac{2s}{n}-2a,\text{ by reducing;}$$

$$\therefore l=\frac{2s}{n}-a,\text{ by collecting the terms.}\quad\text{Q. E. D.}$$

7. The value of a, determined from (2), see Theorem 2d, is $\frac{s}{n}-\frac{n-1}{2}d$; this being substituted for a in (1) gives

$$l=\frac{s}{n}-\frac{n-1}{2}d+\overline{n-1}d.$$

$$\therefore l=\frac{s}{n}+\frac{n-1}{2}d.\qquad\text{Q. E. D.}$$

8. The solution of this is the same as the 4th Theorem up to the last value of $2ds$, with which begin;

$$2ds=l^2-a^2+dl+ad.$$

Hence $l^2+dl=a^2-ad+2ds$, changing sides and transp.

$$l^2+dl+\frac{d^2}{4}=\tfrac{1}{4}\left\{(2a-d)^2+8ds,\right\}\text{ compt. square.}$$

$$l+\frac{d}{2}=\pm\tfrac{1}{2}\sqrt{(2a-d)^2+8ds}, \text{ extracting the root.}$$

$$\therefore l=\pm\tfrac{1}{2}\sqrt{(2a-d)^2+8ds}-\frac{d}{2}. \qquad \text{Q. E. D.}$$

Remark. Since the difference between an increasing and a decreasing series in arithmetical progression consists in the sign of d, and every increasing series may be considered as a decreasing one, which begins with the last term, and has d *minus*, if l be put for a, and a for l, and also $-d$ for $+d$, in the four theorems which express the values of a, we will obtain the corresponding values of l; changing the letters as above, the 1st gives the 5th, the 2d the 7th, the 3d the 6th, and the 4th gives the 8th.

9. In (1), transpose a, and divide by $(n-1)$, and we have

$$d=\frac{l-a}{n-1}. \qquad \text{Q. E. D.}$$

10. From (2), $s=\left\{2a+\overline{n-1}d\right\}\frac{n}{2}$.

$$\therefore 2s=2an+n(n-1)d,$$

or $n(n-1)d=2s-2an$, by transposing, &c.

$$\therefore d=\frac{2s-2an}{n(n-1)}. \qquad \text{Q. E. D.}$$

11. The reduction is the same as that in Theorem 4th, up to the last value of $2ds$, with which begin.

$$2ds=l^2-a^2+ld+ad,$$

$$=l^2-a^2+(l+a)d,$$

$$2sd-(l+a)d=l^2-a^2, \text{ by transposing,}$$

$$\left\{2s-(l+a)\right\}d=l^2-a^2;$$

$$\therefore d=\frac{l^2-a^2}{2s-(l+a)}=\frac{(l+a)(l-a)}{2s-(l+a)}. \qquad \text{Q. E. D.}$$

12. Find a value of $2a$ from (1), which is $2a=2l-2\overline{n-1}d$, and substitute it in the (2), and we will have

$$s=\left\{2l-2\overline{n-1}d+\overline{n-1}d\right\}\frac{n}{2}, \text{ which solve for } d.$$

$$2s=\left\{2l-\overline{n-1}d\right\}n,$$

$$=2ln-n(n-1)d;$$

hence $n(n-1)d=2ln-2s$, by transposition.

$$\therefore d=\frac{2ln-2s}{n(n-1)}=\frac{ln-s}{n-1}\times\frac{2}{n}. \qquad \text{Q. E. D.}$$

13. Replace $a+\overline{n-1}d$ in (2) by its value l in (1),

and we have $s=(a+l)\frac{n}{2}$.

whence $2s=(a+l)n$, by mult. by 2;

and $\therefore n=\frac{2s}{a+l}$. Q. E. D.

---

14. From (1) $a+\overline{n-1}d=l$,

$\therefore a+nd-d=l$,

hence $nd=l-a+d$,

$\therefore n=\frac{l-a}{d}+1$. Q. E. D.

---

15. (2) gives $2s=2an+n^2d-nd$,

$\therefore dn^2+(2a-d)n=2s$, by changing the sides,

Art. (81), $4d^2n^2+4d(2a-d)n+(2a-d)^2=(2a-d)^2+8ds$.

$2dn+2a-d=\pm\sqrt{(2a-d)^2+8ds}$.

$\therefore n=\frac{1}{2d}\left\{\sqrt{(2a-d)^2+8ds}+d-2a,\right\}$ Q. E. D.

16. From (1), $2a=2l-2(n-1)d$; this being substituted for it in (2), gives $s=\left\{2l-\overline{n-1}d\right\}\frac{n}{2}$,

or $2s=2ln-n^2d+nd$,

$dn^2-(2l+d)n=-2s$, by transposition,

$4d^2n^2-4d(2l+d)n+(2l+d)^2=(2l+d)^2-8ds$. (Art. 81.)

$2dn-(2l+d)=\pm\sqrt{(2l+d)^2-8ds}$. Ext. root.

$\therefore n=\frac{2l+d\pm\sqrt{(2l+d)^2-8ds}}{2d}$. Q. E. D.

---

17. In (2), instead of $a+\overline{n-1}d$, substitute l, its value, from (1), and it becomes

$\therefore s=(a+l)\frac{n}{2}=\frac{a+l}{2}n$. Q. E. D.

---

18. This is equation (2) of those from which all the others are derived.

19. Begin with the last value of $2ds$ in the proof of Theorem 4, namely,

$2ds=l^2-a^2+ld+ad$;

$\therefore s=\frac{l^2-a^2}{2d}+\frac{1}{2}(l+a)$. Q. E. D.

---

20. Instead of $2a$ in (2), substitute its value in terms of l, as found from (1), and it becomes

$s=\left\{2l-2\overline{n-1}d+\overline{n-1}d\right\}\frac{n}{2}$,

$\therefore s=\left\{2l-\overline{n-1}d\right\}\frac{n}{2}$. Q. E. D.

Exercises, Page 64.

1. Here $a=5$, $d=4$, and $n=12$; these values being substituted in Theorems 5th and 18th, give

$$l=5+\overline{12-1}\times4=5+11\times4=5+44=49,$$
$$s=\{2\times5+\overline{12-1}\times4\}\tfrac{12}{2}=\{10+44\}6=324.$$

2. Here $a=3$, $l=51$, and $d=2$; these values substituted in Theorems 14th and 19th, will give n and s; thus,

$$n=\frac{l-a}{d}+1=\frac{51-3}{2}+1=\frac{48}{2}+1=25,$$
$$s=\frac{(51)^2-3^2}{4}+\tfrac{1}{2}(51+3)=648+27=675.$$

3. Here $s=12100$, $a=1$, and $d=2$; these values substituted in Theorems 8th and 15th, gives

$$l=\tfrac{1}{2}\sqrt{(2-2)^2+8\times2\times12100}-1=220-1=219,$$
$$n=\tfrac{1}{4}\{\sqrt{(2-2)^2+8\times2\times12100}+2-2\}=\tfrac{1}{4}\sqrt{193600}$$
$$=110.$$

4. Here $n=40$, $a=80$, and $d=4$; these values being substituted in Theorems 5th and 18th, will give

$$l=80+(40-1)\times4=80+156=236,$$
$$s=\{160+39\times4\}20=316\times20=6320.$$

5. Here $a=9$, $d=-\frac{1}{3}$, and $n=21$; substituting these values in Theorem 18th, gives

$$s=\left\{2\times9-20\times\tfrac{1}{3}\right\}\frac{21}{2}=\frac{34}{3}\times\frac{21}{2}=119.$$

6. Here $s=300$, $n=12$, and $d=4$; substituting these values in Theorem 2d, it becomes

$$a=\frac{300}{12}-\frac{12-1}{2}\times4=25-22=3.$$

(96.) GEOMETRICAL PROGRESSION.

$$\left.\begin{array}{|c|l} 1 & l=ar^{n-1}, \\ 2 & s=\dfrac{ar^n-a}{r-1}, \end{array}\right\}$$ Equations previously proved, from which all the others are to be derived.

1. Since $l=ar^{n-1}$; dividing both sides by r^{n-1}, we have

$$a=\frac{l}{r^{n-1}}.$$ Q. E. D.

2. Since by (2) $s=\dfrac{ar^n-a}{r-1}=\dfrac{a(r^n-1)}{r-1}$,

$(r-1)s=a(r^n-1)$, by mult. by $r-1$;

Hence $a=\dfrac{(r-1)s}{r^n-1}$, $\Big\}$ by changing the sides, and dividing by (r^n-1). Q. E. D.

3. Since $l=ar^{n-1}$, $lr=ar^n$; substituting this value of ar^n in (2), we have $s=\frac{lr-a}{r-1}$,

$(r-1)s=lr-a$, by mult. by $r-1$;
$\therefore a=lr-(r-1)s$, by transp. Q. E. D.

4. Since (1) multiplied by r gives $ar^n=lr$; substituting this in (2), we have

$$s=\frac{lr-a}{r-1};$$

Whence $rs-s=lr-a$, by multiplying by $r-1$,
$(s-l)r=s-a$, by transposition;
$\therefore r=\frac{s-a}{s-l}$. Q. E. D.

5. Since (1) $ar^{n-1}=l$,

$r^{n-1}=\frac{l}{a}$, dividing by a.

Extracting the $(n-1)$th root, $r=\sqrt[n-1]{\frac{l}{a}}$. Q. E. D.

6. Is the given equation (2), and 7 is the first equation in the proof of Theorem 4th, which see.

8. From (1) we easily deduce $ar^n=lr$, and $a=\frac{l}{r^{n-1}}$. Substitute these values in (2), and it becomes

$$s=\frac{lr-\frac{l}{r^{n-1}}}{r-1}.$$

Multiply by r^{n-1} both numerator and denominator of the second side, and it becomes

$s=\frac{lr^n-l}{r^n-r^{n-1}}=\frac{l(r^n-1)}{r^n-r^{n-1}}$. Q. E. D.

(9.) Is the given equation (1) previously proved.

10. Beginning with the first expression in the proof of Theorem 4th, we have $s=\frac{lr-a}{r-1}$.

$s(r-1)=lr-a$, mult. by $(r-1)$,
$lr=s(r-1)+a$, transp. and changing sides,
$\therefore l=\frac{s(r-1)+a}{r}$. Q. E. D.

COR. $l=s-\frac{s-a}{r}$.

11. By (2), $s=\frac{ar^n-a}{r-1}=\frac{a(r^n-1)}{r-1}$;

$\therefore a=\frac{(r-1)s}{r^n-1}$; and substituting this value of a in (1), we have $l=\frac{(r-1)s}{r^n-1}\times r^{n-1}$

$$=\frac{r^n-r^{n-1}}{r^n-1}\times s. \qquad \text{Q. E. D.}$$

EXERCISES IN GEOMETRICAL PROGRESSION.

1. Here $a=1$, $r=2$, and $n=10$; therefore
Theorem 9, $l=2^9=2^5\times2^4=32\times16=512$.
Theorem 6, $s=\frac{2^{10}-1}{2-1}=2^5\times2^5-1=32\times32-1=1023$.

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2. Here $a=1$, $l=128$, and $s=255$; therefore substituting these values in Theorem 4th, we have

$$r=\frac{255-1}{255-128}=\frac{254}{127}=2.$$

---

3. Here $a=1$, $r=\frac{1}{2}$, and $n=$ infinity; therefore r^n may be considered as $=0$. Hence (4) equation (Art. 96), gives

$$s=\frac{a}{1-r}=\frac{1}{\frac{1}{2}}=2.$$

---

4. Here $a=1$, $r=\frac{4}{5}$, and $n=$ infinity; therefore

$$s=\frac{1}{1-\frac{4}{5}}=\frac{1}{\frac{1}{5}}=5.$$

---

5. Here $a=m$, $r=\frac{n-1}{n}$, and $n=$ infinity; therefore

$$s=\frac{m}{1-\frac{n-1}{n}}=\frac{m}{\frac{n-n+1}{n}}=\frac{m}{\frac{1}{n}}=mn.$$

---

6. Here $a=1$ farthing, $r=4$, and $n=12$; therefore Theorem 6 gives $s=\frac{4^{12}-1}{4-1}=\frac{(4^4)^3-1}{3}=\frac{(256)^3-1}{3}$

$$=\frac{16777215}{3}\text{ farthings}, = \text{L.}5825,\ 8\text{s. } 5\tfrac{1}{4}\text{d.}$$

7. Let the first term be represented by x, and the common ratio by r, then xr will be the second term, and xr^2 the third; hence

1 | $x+xr+xr^2=52$, Given equations.
2 | $x+xr:x+xr^2::2:5$, Given equations.
3 | $5x+5xr=2x+2xr^2$, by mult. ext. and means of (2).
4 | $5+5r=2+2r^2$, dividing by x.

5	$2r^2-5r=3$, transp. and changing sides.
6	$16r^2-40r+25=49$, completing the square.
7	$4r-5=\pm 7$, extracting.
8	$\therefore r=\frac{5\pm 7}{4}=3$, or $-\frac{1}{2}$.

Substituting the value 3 instead of r in (1), we obtain $13x=52$, $\therefore x=4$; hence we have the numbers sought, 4, 12, and 36.

If the other value of r be substituted, the numbers will be $\frac{468}{7}$, $-\frac{234}{7}$, and $\frac{117}{7}$, which also fulfil the conditions of the question; but one of them being *minus*, they cannot properly be considered as included in the enunciation of the question.

8. Let the notation be the same as in the last exercise; then

	1	$x+xr=15$, By the conditions
	2	$x+xr^2=25$, of the question.
(2)÷(1) gives	3	$\frac{1+r^2}{1+r}=\frac{5}{3}$.
(3)$\times 3(1+r)$	4	$3r^2+3=5r+5$.
(4) transposed	5	$3r^2-5r=2$.
(5) by formula 80	6	$\therefore r=\frac{5\pm 7}{6}=2$, or $-\frac{1}{3}$.
(1) by substitution	7	$x+2x=15$.
(7)÷3	8	$\therefore x=5$, the first.
(8)$\times r$	9	$xr=10$, the second.
(8)$\times r^2$	10	and $xr^2=20$, the third.

(119.)

1. $560\times 3\times \frac{4\frac{1}{2}}{100}=75\frac{6}{10}=$L.75, 12s.

2. $420+420\times 6\times \frac{3}{100}=495\frac{6}{10}=$L.495, 12s.

3. $p=\frac{i}{rt}$(3) $\therefore \frac{60}{\frac{5\times 4}{100}}=\frac{60\times 100}{20}=$L.300.

4. $p=\frac{a}{1+rt}$(3) $\therefore \frac{607\cdot 5}{1+\frac{10\times 3\frac{1}{2}}{100}}=\frac{60750}{135}=$L.450.

5. $t=\frac{a-p}{pr}$(4) $\therefore \frac{800-500}{500\times\cdot 04}=\frac{300}{20}=15$ years.

6. $r=\frac{a-p}{pt}$, $\therefore \frac{344-200}{200\times 18}=\frac{144}{3600}=\frac{4}{100}=4$ per cent.

(121.)

1. By Formula (1), Log. 1000 = 3·0000000000
By Table 12; 10 Log. 1·05 = 0·2118929910
Ans. L.1628, 17s. 9$\frac{1}{2}$d. = 1628·89 = 3·2118929910

2. By Formula (2), Log. 700 = 2·845098
11 Log. 1·04 = 0·1873667323
Ans. L.454, 14s. 1$\frac{1}{2}$d. = 454·706 = 2·657731

3. By Formula (4), $\frac{2\cdot602060-2\cdot562293}{\cdot0170333393}$ = 2 yrs. 122 days.

4. By Formula (5), $\frac{\text{Log. } 63\cdot81354-\text{Log. } 50}{5}=$

$\frac{1\cdot804913-1\cdot698970}{5}=\cdot021189=$ Log. 1·05;

therefore the rate per cent is 5. Ans.

5. Apply Formula (4). This question is the same as,— In what time will one pound become two, at 5 per cent? hence $\frac{\text{Log. } 2-\text{Log. } 1}{\text{Log. } 1\cdot05}=\frac{\cdot301030}{\cdot021189}=14\cdot2$ years.

6. $\frac{\text{Log. } 2-\text{Log. } 1}{\text{Log. } 1\cdot04}=\frac{\cdot301030}{\cdot017033}=17\cdot67$ years.

7. $\frac{\text{Log. } 100-\text{Log. } 10}{\cdot021189}=\frac{1}{\cdot021189}=47\cdot19$ years.

8. Here the amount must first be found (1). And the value of R will be found by (Art. 121); we have then,
Log. 100 + 12, Log. 1·04 = Log. 160·103 = L.160, 2s. 0$\frac{3}{4}$d.
Log. 100 + 24, Log. 1·02 = Log. 160·844 = L.160, 16s. 10$\frac{1}{2}$d.
Log. 100 + 48, Log. 1·01 = Log. 161·223 = L.161, 4s. 5$\frac{1}{2}$d.

Subtracting the principal, L.100, from each of the above, we obtain the compound interest sought.

(127.)

1. Apply Formula, (Art. 125), and $\frac{a}{r}=2000, \left(1-\frac{1}{(1+r)^n}\right)=\cdot623111$; hence $2000\times\cdot623111=1246\cdot222=$ L.1246, 4s. 5$\frac{1}{4}$d.

2. Apply the last formula in (Art. 125), viz. $p=\frac{a}{r}$; hence the value sought is $\frac{100}{\frac{5}{100}}=100\times\frac{100}{5}=$ L.2000.

3. This is the same as finding the present value of an annuity of L.50 for 40 years, at 2$\frac{1}{2}$ per cent, to which apply formula, (Art. 125), viz. $p=\frac{a}{r}\left(1-\frac{1}{(1+r)^n}\right), \frac{a}{r}=2000$ $(1-\cdot372431)=2000\times\cdot627569=$ L.1255, 2s. 9$\frac{1}{4}$d.

4. *This is the same* as finding the value of a perpetual

annuity of L.50 per annum at one-half of 5 per cent; and hence we have, by substituting these values in the last formula, (Art. 125),

$$p=\frac{50}{\cdot 025}=\frac{50000}{25}=\text{L.}2000.$$

5. By (Art. 126), and substituting the proper values of t, n, and r, we have

$$p=\frac{\{(1\cdot 04)^{30}-1\}}{\cdot 04(1\cdot 04)^{40}}\times 100=2500\left\{\frac{(1\cdot 04)^{30}-1}{(1\cdot 04)^{40}}\right\}=2500\times$$

$$\frac{2\cdot 243398}{4\cdot 801021}=\text{L.}1168,\ 3\text{s. }9\text{d.}$$

6. Here $p=\dfrac{(1\cdot 05)^{42}-1}{(1\cdot 05)^{50}}\times\dfrac{50}{\cdot 05}=\dfrac{6\cdot 76156}{11\cdot 4674}\times\dfrac{5000}{5}=$
$\dfrac{6761\cdot 59}{11\cdot 4674}=589\cdot 636=$L.589, 12s. 8½d.

7. By (Art. 127), calling the amount a, we have

$$a=\frac{A}{r}\{(1+r)^n-1\};$$

$\therefore\dfrac{ra}{A}+1=(1+r)^n$, by mult. by $\dfrac{r}{A}$ and transp.

or Log. $\left\{\dfrac{ra}{A}+1\right\}=n$ Log. $(1+r)$. But $\left(\dfrac{ra}{A}+1\right)=2$;

$\therefore\dfrac{\text{Log. }2}{\text{Log. }(1\cdot 05)}=n$, or $\dfrac{\cdot 301030}{\cdot 021189}=14\cdot 2$.

8. In Formula, (Art. 127), put $A=24$, $r=\cdot 05$, or $\frac{1}{20}$, $(1+r)=1\cdot 05$, and $n=20$, then $\dfrac{A}{r}=480$, and $\{(1+r)^{20}-1\}$ $=1\cdot 6533$, which being multiplied by 480, gives $793\cdot 584$; therefore the amount sought is L.793, 11s. 8d.

Note. The exercises in compound interest may be easily solved by Tables 7th and 8th of this work; and those in annuities by Tables 9th and 10th.

Promiscuous Exercises. Page 78.

(1.) Let x be the first part of 11, then $11-x$ will be its second part; also let y be the first part of 17, then $17-y$ will be its second part.

	1	$xy=45,$	By the question.
	2	$187-11y-17x+xy=48,$	
(2)—(1)	3	$187-11y-17x=3.$	
(3) transp.	4	$11y=184-17x.$	
(4)÷11	5	$y=\dfrac{184-17x}{11}.$	
(1) by substitution	6	$\dfrac{184x-17x^2}{11}=45.$	

(6) × 11	7	$184x-17x^2=495.$
(7) signs changed	8	$17x^2-184x=-495.$
(8) by Art. 80	9	$34x=184\pm14=198$, or 170.
(9) ÷ 34	10	$\therefore x=5.$
(1) by substitution	11	$5y=45.$
(11) ÷ 5	12	$\therefore y=9.$
By the question	13	$11-x=11-5=6.$
	14	$17-y=17-9=8.$

(2.) Let x represent the first part of 30; then $3x$ will represent the first part of 21; and therefore the second part of 21 will be represented by $21-3x$, and the second part of 30 by $30-x$; hence by the question,

$$(21-3x)^2+(30-x)^2=585.$$

hence $441-126x+9x^2+900-60x+x^2=585.$

$10x^2-186x=-756$, by collecting and transp.

$5x^2-93x=-378$, by dividing by 2.

$100x^2-1860x+8649=1089$, by completing the square.

$10x-93=\pm33$, by extracting the root.

$$\left.\begin{aligned}\therefore x&=6,\\ 30-x&=30-6=24,\\ 3x&=18,\\ \text{and } 21-3x&=21-18=3,\end{aligned}\right\}\text{Values sought.}$$

(3.) Let x represent the sum left to the first; then since the shares were in continued proportion, if r be put for the common ratio, the second's share will be rx, and the third's r^2x.

	1	$x+rx+r^2x=210,$ By the question.
	2	$x-r^2x=90,$
(1) ÷ (2)	3	$\frac{1+r+r^2}{1-r^2}=\frac{210}{90}=\frac{7}{3}.$
(3) × $3(1-r^2)$	4	$3+3r+3r^2=7-7r^2.$
(4) transposed	5	$10r^2+3r=4.$
	6	$400r^2+120r+9=169$, by completing square.
	7	$20r+3=13$, by extracting the root.
(7) transposed	8	$r=\frac{1}{2}$, and divided by 20.
(2) by substitution	9	$x-\frac{1}{4}x=90.$
(9) × by 4	10	$3x=360.$
	11	$x=120$; hence $rx=60$, and $r^2x=30$.

(4.) Let x represent the greater of the two numbers; then since their product is 45, the other will be $\frac{45}{x}$; and by the question we have

$x^2-\frac{2025}{x^2}:\left(x-\frac{45}{x}\right)^2=7:2$. mult. extr. and means.

$2x^2-\frac{4050}{x^2}=7x^2-630+\frac{14175}{x^2}$.

$2x^4-4050=7x^4-630x^2+14175$, by × by x^2.

$5x^4-630x^2=-18225$, by transposition.

$x^4-126x^2=-3645$, by dividing by 5.

$x^4-126x^2+3969=324$, by completing the square.

$x^2-63=\pm 18$, by extracting the root.

$x^2=63+18=81$; the other value is not a square.

$\therefore x=9$.

hence $\frac{45}{x}=\frac{45}{9}=5$.

5. Let x represent A's capital, then $100-x$ will represent B's; also let y represent the rate of gain per pound per month; then by the question,

$x+3xy=100-x+2y(100-x)$.

$x+3xy=100-x+200y-2xy$.

$5xy-200y=100-2x$, by transposition,

$(5x-200)y=100-2x$.

$$\therefore y=\frac{100-2x}{5x-200}.$$

But by the question, $x+3xy=99$; hence substituting the above value of y, we have

$x+3x\left(\frac{100-2x}{5x-200}\right)=99$.

$5x^2-200x+300x-6x^2=495x-19800$.

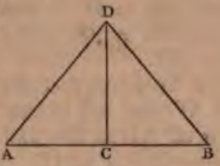
$x^2+395x=19800$, transp. and changing the signs.

$x^2+395x+\frac{156025}{4}=\frac{235225}{4}$.

$x+197\frac{1}{2}=242\frac{1}{2}$, extracting the root.

$\left.\begin{array}{l}\therefore x=45,\\ \text{and } 100-x=100-45=55,\end{array}\right\}$ The values sought.

GEOMETRICAL EXERCISES, (P. 152.)

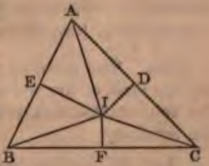


1. Let the | DC bisect the | AB at $r^t\angle s$, every point in the line DC is equally distant from the points A and B.

For take any point, as D in DC, and join DA and DB, then the $\triangle s$ ACD, BCD, have AC$=$CB, and CD common, and the contained $\angle s$ ACD, BCD, also equal; $\therefore$ (Prop. 5) the base AD is $=$BD, and hence the point D is equally distant from the points A and B; and the same may be proved of any point in the line DC; $\therefore$ every point in DC is equally distant from A and B.

Q. E. D.

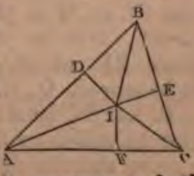
Cor. The locus of all points that are equally distant from two given points, is the line which bisects their distance at right angles.



2. Let |*s* EI, DI, be drawn, bisecting the two sides BA, AC, of the triangle ABC, at $r^t\angle s$; the straight line IF drawn from the point of their intersection I, $\perp$ to the third side BC, will bisect it.

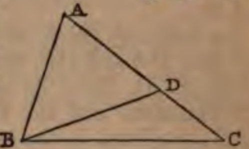
For join AI, BI, CI; then the $\triangle s$ AEI, BEI, have BE$=$AE, and EI common to both, and the $\angle$BEI is $=$ the $\angle$AEI, each being a $r^t\angle$; $\therefore$ (Prop. 5) the base BI is $=$AI; in the same manner it can be proved from the $\triangle s$ ADI, CDI, that AI is $=$CI. Hence, since each of the lines BI, CI, is equal to AI, they are $=$ one another; $\therefore$ BIC is an isosceles triangle, and the $\angle s$ IBC, ICB, are equal, (Prop. 7); $\therefore$ since the angles at F are $r^t\angle s$, the $\triangle s$ IBF, ICF, have two angles in the one $=$ two angles in the other, and the side IF opposite to the equal $\angle s$ IBF, ICF, is common to both; $\therefore$ BF is $=$CF, (Prop. 19, cor. 5); hence the third side is bisected by the $\perp$ IF.

Q. E. D.



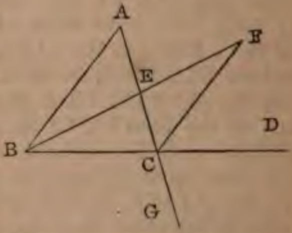
3. Let the |*s* AI, CI, bisect the $\angle s$ BAC, BCA, then the |BI drawn from their point of intersection to the third $\angle$B, will bisect it. Draw ID, IE, IF, $\perp$ to AB, BC, AC; then in the $\triangle s$ DAI, FAI, the $\angle s$ DAI, FAI, are equal by construction, *the* $\angle s$ *at* D *and* F *are* $r^t\angle s$, *and the side AI opposite* the r^t $\angle s$ is common to both;

∴ (Prop. 19, cor. 5) DI is =FI; in the same manner, in the △s ECI, FCI, it can be proved that EI is =FI, and ∴ DI is =EI; and since BI is common to △s BDI, BEI, and the ∠s at D and E are $r^t\angle$s, (Prop. 39, cor. 2), the ∠DBI is = the ∠EBI; ∴ the third angle is bisected by the |BI. Q. E. D.

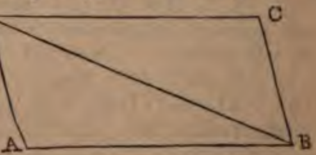


4. To prove that the difference of AC and AB is less than BC, cut off AD=AB, and join BD. Since AD is =AB, the ∠ABD is = the ∠ADB; ∴ (Prop. 19) each of them is less than a $r^t\angle$; but the two ∠s ADB, BDC, are together = two right ∠s; ∴ BDC is > a $r^t\angle$, and hence (Prop. 19), the ∠DBC is < a $r^t\angle$, and ∴ less than the ∠BDC. And since in the △BDC the ∠BDC is > the ∠DBC, the side BC is > the side DC; but DC is the difference of AC and AB, and BC is the third side; ∴ the difference of the sides AC, AB, is < BC; in the same manner it might be proved, that the difference of any other two sides is less than the third. Q. E. D.

Note. The proof of this proposition may be obtained by producing the least of the two sides, till it be equal to the greater, &c. Or, the demonstration may be seen, as deduced from Proposition 13, in the corollary.



5. Let AC be the base; bisect it in E, join BE, and produce it to F, making EF=BE, and join FC; then we have to prove that AB+BC is >2BE, that is, than BF. Now since AE is =EC, and BE is =EF, the two sides AE, EB, are = the two sides CE, EF, and the contained ∠s AEB, CEF, are also = (Prop. 3); ∴ the side AB is = the side CF, (Prop. 5). To each of these equals add BC, and AB+BC will be =FC+CB; but FC+CB is >BF, (Prop. 13); ∴ also AB+BC is >BF, that is, than twice BE, which was to be proved. Q. E. D.



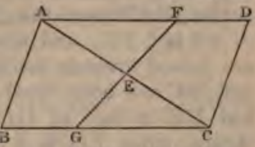
6. Let DC be =AB, *and AD=BC*, we have *to prove that* the figure *ABCD is* a parallelo-

gram. ∵ in the △s DAB, BCD, DC is =AB, DA=CB, and the side DB common; ∴ (Prop. 9) the ∠s are equal that are opposite to the equal sides; hence the ∠ABD is = the ∠CDB, and (Prop. 16) AB is ∥ DC; also the ∠ADB is =∠CBD, and AD is ∥ CB; ∴ since AB is ∥ CD, and AD is ∥ CB, the figure ABCD is a parallelogram.

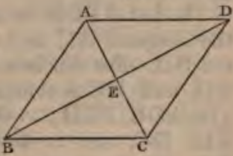
Q. E. D.

7. In the figure to last exercise, let the ∠DAB be = the ∠BCD, and the ∠ADC be = the ∠CBA, it is required to prove that the figure ABCD is a parallelogram. Since the figure can be divided into two triangles, having their angles together equal to the angles of the quadrilateral figure; ∴ all the angles of the quadrilateral figure are together = four r^t∠s, and ∴ any two adjacent angles, as DAB, ABC, are together equal to two right angles, hence (Prop. 16, cor. 2), AD, BC, are parallel; and in the same manner it may be proved, that AB and DC are parallel, and ∴ by (Def. 38) the figure is a parallelogram.

Q. E. D.

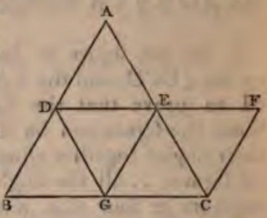


8. Let ABCD be a ▱, and AC its diagonal, any straight line, as FG, that bisects the diagonal AC, also bisects the ▱. Since ABCD is a ▱, and AC its diagonal, the △ABC is = the △CDA, (Prop. 34), and the △s AEF, CEG, have the ∠s AEF, CEG, equal, (Prop. 3), and the ∠s AFE, EGC, equal, (Prop. 17), and also the side AE=CE, which are opposite to equal ∠s; ∴ the △AEF is = the △CEG, (Prop. 19, cor. 5); from the △ABC, which is half of the ▱, take away the △GEC, and add the equal △FEA, then it is evident that the figure ABGF is half of the ▱ABCD, and therefore the ▱ is bisected by the |FG, which bisects its diagonal. Q. E. D.

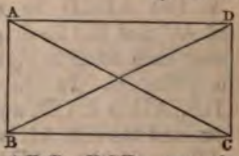


9. Let ABCD be a rhombus, and AC and BD its diagonals, they bisect each other at right angles in the point E. In the △s ADB, CDB, AD is =CD, and BD common, and the side AB is also =BC; ∴ (Prop. 9) the ∠ADB is = the ∠CDB. Again, in the △s ADE, CDE, the two sides AD, DE, are = the two sides CD, DE, and the contained angle ADE is also = the contained ∠CDE; ∴ the base AE is = the base EC, and the ∠AED to the ∠CED,

(Prop. 5), and since they are adjacent angles, each of them is a $r^t\angle$, (Def. 9.) And since AE is =EC, AC is bisected in E, and in the same manner it may be proved that BD is bisected in E; ∴ the diagonals of a rhombus bisect each other at $r^t\angle s$. Q. E. D.



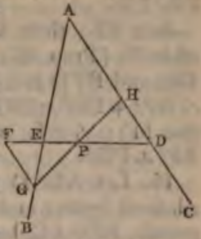
10. Let ABC be a △, whose sides AB, AC, are bisected in D and E, then the line DE is ∥ BC, is = the half of BC, and cuts off a △ADE = the fourth part of the △ABC. Produce DE to F, so that EF may be equal to DE, and join FC, and through E draw EG ∥ AB, and join DG; in the △s AED, CEF, the two sides AE, ED, are = the two CE, EF, and the contained angle AED is = the contained angle CEF; ∴ the base AD is =CF, but AD is =DB; ∴ DB is =FC, and the remaining angles are equal to which the equal sides are opposite; ∴ the ∠EAD is = the ∠ECF, and they are alternate ∠s; ∴ AB is ∥ FC; and since DB is = and ∥ FC, and they are joined towards the same parts by DF and BC, ∴ DF and BC are both equal and parallel, (Prop. 24, cor. 1); but DE is half of DF, hence also DE is half of BC. Again, since EG is ∥ DB, and DE is ∥ BG, the figure DBGE is a ▱; ∴ BG is =DE, (Prop. 24), and hence G is the middle of the base; and since DG bisects the two sides BA, BC, it is ∥ AC, and each of the figures ADGE, DGCE, is a ▱. and DBGE was proved to be a ▱; ∴ the △s ADE, DBG, and EGC, are each = the △DGE, and hence the triangle ABC is thus divided into four equal △s, ADE, DBG, GDE, and EGC, and ∴ the △ADE, cut off by the line DE, is a fourth part of the △ABC. Q. E. D.



11. Let ABCD be a $r^t\angle d$ ▱, the diagonal AC is = the diagonal BD. For AD is =BC, (Prop. 24), and DC is common to both △s, ADC, BCD; ∴ the two sides AD, DC, are = the two sides BC, CD, and the contained ∠s ADC, BCD, are also equal; ∴ the base AC is = the base BD, (Prop. 5), and the diagonals of the rectangle are equal to one another.

Q. E. D.

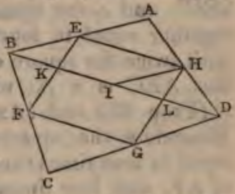
12. Let P be the given point, and AB, AC, the given straight lines, it is required to draw through the point P a straight line, which shall be terminated in the |s AB, AC, and bisected in the point P.



Draw any |ED through the point P, and produce it to F, making PF=PD, and through F draw FG ∥ AC, till it cut AB in G; join GP, and produce it to meet AC in H; GPH is the line required.

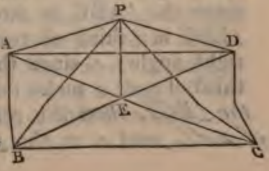
Since FG is ∥ AD, and DF meets them, the ∠GFP is = the ∠HDP, and the ∠FPG is = the ∠DPH, (Prop. 3); ∴ since the side FP is = the side PD, (Prop. 6), the side GP is = the side HP; hence the line GH has been drawn, terminated in AB, AC, and bisected in the point P, which was required to be done.

13. Let ABCD be a quadrilateral figure, whose sides are bisected in the points E, F, G, and H, the figure EFGH, formed by joining the adjacent middle points, is a parallelogram, whose area is half of the quadrilateral figure ABCD.



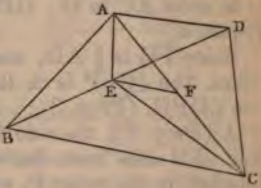
Join BD; then by Exercise 10, ∵ ABD is a △ whose sides AB, AD, are bisected in E and H, EH is ∥ BD, and = half of it; for a similar reason FG is ∥ BD, and = half of it; ∴ EH and FG are two equal and parallel straight lines, and they are joined towards the same parts by the |s EF and HG; ∴ (Prop. 24, cor. 1) EF and HG are themselves equal and parallel, and the figure EFGH is a parallelogram. Draw HI parallel to AB; then by Exercise 10 the △s AEH and HID are each a fourth part of the △ABD, and are ∴ together half of the whole triangle ABD, and hence the remaining ▱EBIH is half of the △ABD; but the ▱EBIH is = the ▱EKLH, (Prop. 26), being on the same base EH, and between the same parallels EH and BD. For a like reason the ▱KFGL is half of the △CBD; ∴ the whole ▱EFGH is half of the whole quadrilateral figure ABCD. Q. E. D.

14. Let ABCD be a rectangle, and P any point from which there are drawn PA, PC, to the extremities of the diameter AC, and PB, PD, to the extremities of the *diagonal BD;* the sum of the



squares of PA, PC, is equal to the sum of the squares PB, PD.

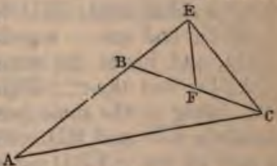
Join PE; then APC is a △, whose base, AC, is bisected in E, (Prop. 45): ∴ $AP^2+PC^2=2AE^2+2EP^2$, (Prop. 43), and BPD is a △, whose base, BD, is bisected in E; ∴ $BP^2+PD^2=2BE^2+2EP^2$. But AC is = BD, (Exercise 11); ∴ AE is = BE; wherefore AP^2+PC^2 is = BP^2+PD^2. Q. E. D.



15. Let ABCD be a quadrilateral figure, and let its diagonals, BD, AC, be bisected in the points E and F, and join EF, EA, and EC; we have to prove that the squares of AB, BC, CD, and DA, are together equal to the squares of AC and BD, and four times the square of EF.

Because BAD is a △, whose base, BD, is bisected in E, the squares of BA, AD, are $=2BE^2+2AE^2$, (Prop. 43); for a like reason, the squares of BC, CD, are $= 2BE^2+2EC^2$, and ∴ the squares of BA, AD, DC, and CB, are together equal to four times the square of BE, together with twice the square of AE and twice the square of EC. But AEC is a △, whose base, AC, is bisected in F; ∴ (Prop. 43) and (Ax. 6) $2AE^2+2EC^2=4AF^2+4EF^2$. Wherefore the squares of AB, BC, CD, and DA, are equal to four times the square of BE, four times the square of AF, and four times the square of EF; but (Prop. 37. cor. 2) four times the square of BE is = the square of BD, and four times the square of AF is = the square of AC; ∴ $AB^2+BC^2+CD^2+DA^2=BD^2+AC^2+4EF^2$.

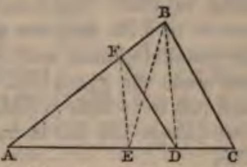
Q. E. D.



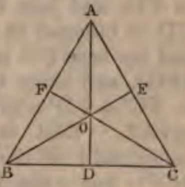
16. Let ABC be a triangle, having the angle, ABC, two-thirds of two right angles, the square on AC will be equal to the squares on AB, BC, together with the rectangle contained by AB, BC. Produce AB to E, and from C draw CE perpendicular to AB; then since the ∠ABC is two-thirds of two right angles, the ∠EBC is a third of two right angles, or two-thirds of a right angle; ∴ since the ∠ at E is a r^t∠, the ∠ECB is a third of a r^t∠; make the ∠CEF a third of a r^t∠, and ∴ = the ∠ECF, then the remaining ∠BEF will be two-thirds of a r^t∠, and ∴ = the ∠EBF, ∴ also the ∠EFB is two-

thirds of a $r^t\angle$, hence the △EBF is equilateral; and since the ∠FEC is = the ∠FCE, the side FE is = the side FC; ∴ BE is the half of BC. Again, since ABC is an obtuse angled △, the square of AC is = the squares of AB and BC, together with twice the rectangle $AB\cdot BE$, (Prop. 40); but twice AB into BE is = AB into BC; ∴ $AC^2 = AB^2 + BC^2 + AB\cdot BC$. If the ∠ABC were two-thirds of one $r^t\angle$, the perpendicular, CE, would fall within the △, and would cut off a part from BA = the half of BC, so that by Proposition 41, the square of AC would be less than the squares of AB and BC by the rectangle $AB\cdot BC$.

Q. E. D.

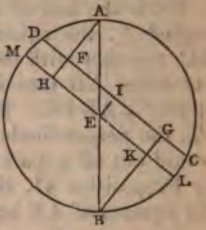


17. Let ABC be the given triangle, and D the given point, it is required to draw from the point D a straight line, which shall bisect the △ABC. Bisect AC in E, and join BE, BD, and through E draw EF ∥ BD, and join DF, the △ABC will be bisected by the |DF. ∵ AE is = EC, the △AEB is = the △BEC, (Prop. 26), ∴ the △s AEF, FEB, are together half of the △ABC; but the △FEB is = the △FED, ∵ they are upon the same base FE, and between the same parallels FE, BD; wherefore the △s AEF, FED, are together half of the whole △ABC, and the triangle ABC is ∴ bisected by the |DF, drawn from the point D, which was required to be done.



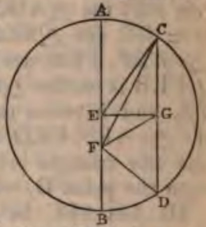
18. Let ABC be an equilateral triangle, and AD a line drawn ⊥ to BC, since in the △s ABD and ACD, the ∠s at D are $r^t\angle$s, and the ∠s ABD, ACD, are equal; ∴ the ∠s BAD, CAD, are equal, (Prop. 19); and since the ∠BAC is two-thirds of a $r^t\angle$, each of them is a third of a $r^t\angle$. Again, if CF be drawn ⊥ to AB, it can be proved in the same manner that it bisects the ∠ACB, and by Exercise 3 the | drawn through the points B and O will bisect the ∠ABC, and (Prop. 7, cor. 2) the sides AB, BC, CA, are bisected in the points F, D, and E. The | OD, OE, OF, are all equal by Exercise 3, and being ⊥ to the sides, are the radii of the inscribed circle. But it was proved in Exercise 16, that when a right-angled △ has one of its angles a third of a $r^t\angle$, the side opposite to that ∠ is half of the hypotenuse; ∴ AO is double of OE or OD; to each add

OD, and we have $AD=3OD=$ three times the radius of the inscribed circle. Q. E. D.



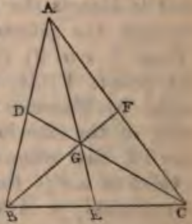
19. Let ADBC be a circle, of which AB is the diameter, and DC is any chord on which there are drawn AF and BG, perpendiculars from the extremities of the diameter AB; it is required to prove that DF is $=$ CG.

Through the centre E draw the diameter LM parallel to the chord CD, and produce AF to meet it in H, and through E draw EI perpendicular to DC. Since each of the lines AF, BG, and EI, are ⊥ to CD, they are parallel, and the figures HI, EG, are parallelograms, and ∴ FI is $=$ HE, and IG is $=$ EK; and since the △s AEH, BEK, are evidently equiangular, and AE is $=$ EB, ∴ EH is $=$ EK, and hence IF is $=$ IG. But (Prop. 46) ID is $=$ IC, ∴ the remainder, DF, is $=$ CG. Q. E. D.



20. Let ABD be a circle, of which AB is the diameter, and CD a chord parallel to AB, it is required to prove, that if a point F be taken in the diameter, and |s FC, FD be drawn from it to the extremities of the chord CD, that $FC^2+FD^2=FA^2+FB^2$.

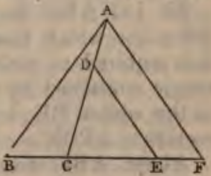
Through the centre E draw EG ⊥ to CD, and join FG and EC; then CD is bisected in G, (Prop. 46). And ∵ CFD is a △, and its base CD is bisected in G, $FC^2+FD^2=2CG^2+2FG^2$, and since GEF is a r^t∠, FG^2 is $= FE^2+EG^2$; ∴ $FC^2+FD^2=2CG^2+2EG^2+2EF^2$; but $CG^2+EG^2=EC^2$, ∵ EGC is a r^t∠, (Prop. 39), and EC is $=$ AE, being radii of the same circle, ∴ $FC^2+FD^2=2AE^2+2EF^2$; but ∵ the line AB is bisected in E, and divided unequally in F, $2AE^2+2EF^2$ is $= AF^2+FB^2$, (Prop. 38, cor. 2), ∴ the squares of FC and FD are equal to the squares of AF and FB. Q. E. D.



21. Let ABC be a triangle, and from its angular points B and C let there be drawn to the middle of the opposite sides the |s BF and CD, and through *their* point of intersection G, and the *third angle* at *A*, draw the |AE; it is

required to prove that BG is to GF, as also CG to GD, as two is to one, that the base BC is bisected by the |AE, and that the six △*s* ADG, GDB, BEG, GEC, CGF, and FGA, are all equal. Because AB and AC are bisected in the points D and F, each of the △*s* ADC, AFB, is half of the whole △ABC, and ∴ they are equal to one another. From each take the common part ADGF, and there remains the △GDB = the △GFC; but since AD=DB, the △BDG is = the △ADG, and since AF=FC, the △CFG is = the △AFG, ∴ the △ADG is = the △AFG; and since the △ABG is double of the △ADG, it is also double of the △AFG, and ∴ the △ABG is to the △AFG as two is to one. But since the △*s* ABG and AFG have the same altitude, they are to one another as their bases, (Prop. 58); ∴ from equality of ratios, BG is to GF as two is to one. Again, since the △BDG is = the △CFG, and the △ABG is double of the △BDG, and the △ACG is double of the △CFG, ∴ the △ABG is = the △ACG, and since they are on the same base AG, they have the same altitude, (Prop. 29, schol.); but the altitude of these △*s* is the same as that of the △*s* GBE and GCE, which being on the same base GE, are ∴ equal. And since the △*s* GBE and GCE are equal, and lie between the same parallels, they are on equal bases, BE, EC, and therefore the third side, BC, is bisected by the |AE, and the six △*s* ADG, GDB, BEG, GEC, CGF, and FGA, are all equal.

Q. E. D.



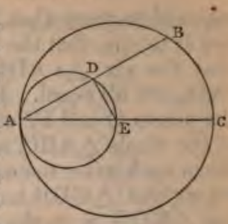
22. Let ABC and DCE be two triangles, having the ∠ABC = the ∠DEC, and the ∠*s* ACB, DCE, together equal to two right angles; it is required to prove that the sides about the third angles BAC and EDC are proportional, so that BA : AC=ED : DC.

Through A draw AF ∥ DE, meeting BE produced in F; then because AF is ∥ DE, the angle DEC is = the angle AFC, (Prop. 17); but the ∠DEC is = the ∠ABC by hypothesis, ∴ the ∠ABF is = the ∠AFB, and hence the side AB is = AF, (Prop. 8.) Again, ∵ the △*s* ACF, DCE, are evidently equiangular, FA : AC=ED : DC, (Prop. 61), and since FA is = BA, ∴ BA : AC=ED : DC.

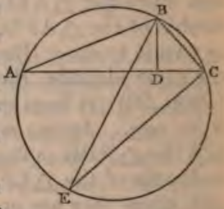
Q. E. D.

23. Let ABC be a circle, and ADE another, described on the radius of the *former* as its diameter; any chord AB *drawn in the former* from the point of contact, will be

bisected by the latter. From the centre E of the circle ABC, draw ED, to the point where the straight line AB cuts the inner circle ADE, then the $\angle$ADE being in a semicircle, is a right angle, (Prop. 52); and since ED is drawn from the centre E at right angles to the chord AB, ∴ (Prop. 46) AB is bisected in D. Q. E. D.



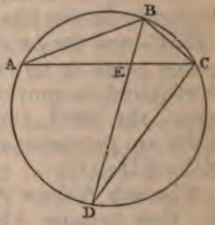
24. Let ABC be a triangle, and BD a perpendicular, drawn from B, upon the third side AC; it is required to prove that the rectangle AB·BC is equal to the rectangle contained by BD, and the diameter of the circle described about the triangle ABC.



About the $\triangle$ABC, describe the circle ABCE, and draw the diameter BE, and join CE, then the $\angle$BCE being in a semicircle, is a right angle, and ∴ equal to the $\angle$ADB; but the $\angle$BEC is also equal to the $\angle$BAD, ∵ they are in the same segment of a circle, or stand on the same arc BC; ∴ the $\triangle$s ABD and EBC are equiangular, (Prop. 19, cor. 1), and hence (Prop. 61) $AB : BD = EB : BC$, ∴ (Prop. 64, cor. 2) the rectangle AB·BC is = the rectangle EB·BD.

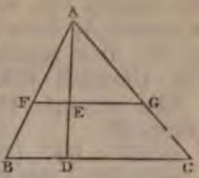
Q. E. D.

25. Let ABC be a triangle, and BE a line which bisects the $\angle$ABC; it is required to prove that the rectangle contained by AB·BC is equal to the square BE, together with the rectangle AE·EC.

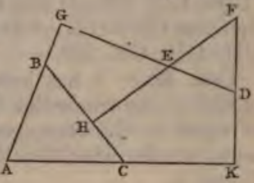


About ABC describe the circle ABD, and produce BE, to meet the circumference in D, and join DC; then the $\angle$s ABE, DBC, are equal by construction, and the $\angle$BAE, BDC, are equal, because they stand on the same arc BC, (Prop. 47, cor. 1); ∴ the $\triangle$s ABE, BDC, are equiangular, (Prop. 19, cor. 1), and hence $AB : BE = DB : BC$. ∴ $AB \cdot BC = DB \cdot BE$; but $DB \cdot BE$ is $= BE^2 + BE \cdot ED$, (Prop. 36), and $BE \cdot ED$ is $= AE \cdot EC$, (Prop. 65), and ∴ $AB \cdot BC$ is $= BE^2 + AE \cdot EC$. Q. E. D.

26. Let ABC be a triangle, and FG a straight line *drawn parallel to* BC, one of its sides, and AD a line *drawn from A, cutting* the parallels BC, FG; it is requir-



ed to prove that BD : DC = FE : EG. The △s ADB, AEF, are manifestly equiangular, as also the △s ADC, AEG; ∴ AD : DB = AE : EF, and alternately AD : AE = DB : EF; also in the △s ADC, AEG, since they are equiangular, AD : DC = AE : EG, (Prop. 61), and alternately AD : AE = DC : EG; but it was proved that AD : AE = DB : FE, ∴ DB : EF = DC : EG, (Alg. 102), and alternately BD : DC = FE : EG. Q. E. D.

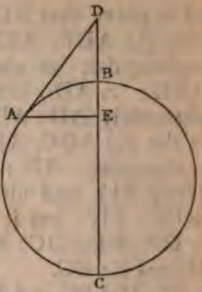


27. If ABC and DEF be two triangles, which have their sides respectively perpendicular to one another, namely, AB ⊥ to ED, BC ⊥ to EF, and AC ⊥ to FD; it is required to prove that the △ABC is equiangular to the △DEF, and ∴ similar to it.

Produce the sides AB and DE to meet in G, AC and DF to meet in K, and produce FE to meet BC, produced if necessary in H; then each of the angles at G, H, and K, are right angles, since the sides are respectively perpendicular to one another. Since EGBH is a quadrilateral figure, its four angles are together equal to four right angles, (Prop. 22, cor. 1); but the ∠s at G and H are right angles, therefore the ∠s GBH and HEG are together = 2 r^t ∠s; but GBH and CBA are also together = 2 r^t ∠s, ∴ the two ∠s GBH and HEG are together = the two ∠s GBH and CBA; take away the common ∠GBH, and there remains the ∠HEG = the ∠CBA; but the ∠HEG is = the ∠DEF, (Prop. 3), ∴ the ∠ABC is = the ∠DEF. Again, in the quadrilateral figure FHCK, since the ∠s at H and K are r^t ∠s, the remaining ∠s HCK and HFK are together = 2 r^t∠s; but the ∠s HCK and BCA are also together = 2 r^t ∠s, ∴ the ∠s HCK and HFK are together = the two ∠s HCK and BCA; take away the common ∠HCK, and there remains the ∠ACB = the ∠DFE; hence also the third ∠ at A is equal to the third ∠ at D, which might also be proved in the same manner as the other two, from the quadrilateral figure GAKD; ∴ the triangles ABC, DEF, which have their sides respectively perpendicular to one another, are equiangular, and therefore similar.

Q. E. D.

28. Let ABC *be a circle,* and BC its diameter, and D *a point in the diameter* produced, from which there is drawn

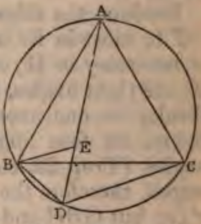


the tangent DA, and from the point of contact A, draw AE perpendicular to BC; it is required to prove that CD : DB = CE : EB.

The rectangle CD·DB is $= AD^2$, (Prop. 67,
$= AE^2 + DE^2$, (Prop. 39),
$= CE\cdot EB + ED\cdot EB + ED\cdot DB$.
(Props. 68 and 35.)

From each of these equals take the rectangle ED·DB, and there remains $(CD - ED)\cdot DB = (CE + ED)EB$, and therefore $CE\cdot DB$ is $= CD\cdot BE$; and hence (Prop. 64, cor. 2) CD : DB = CE : EB. Q. E. D.

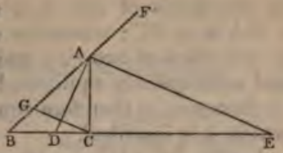
Cor. Since it has been proved that the rectangle contained by the whole line CD, and the middle segment EB, is equal to the rectangle contained by the two extreme segments, the line CD is divided harmonically in the points E and B.



29. Let ABC be an equilateral triangle inscribed in a circle, and D a point in the circumference, from which there are drawn the three straight lines DB, DC, DA; it is required to prove that AD is equal to the sum of DB and DC. From DA cut off DE = DB, and join BE; then ∵ DE is = DB, the $\angle$DBE is = the $\angle$DEB; but the $\angle$BDE is = the $\angle$ACB, (Prop. 47, cor. 1), and therefore a third of two right angles, and hence each of the remaining angles is a third of two right angles, and the triangle EBD is ∴ equilateral, and the side EB is = BD. Again, the $\angle$DBE is = the $\angle$ABC, for each is a third of two $r^t\angle s$; from each take the $\angle$EBC, and there remains the $\angle$DBC = the $\angle$EBA, also the $\angle$BCD is = the $\angle$BAE, (Prop. 47, cor. 1), and the side BC is = the side BA, being sides of an equilateral triangle; ∴ the $\triangle s$ CBD, ABE, have two angles of the one equal to two angles of the other, and a side lying between these equal angles also equal; ∴ these triangles are equal in every respect, (Prop. 6), and hence AE is = CD; and by construction, ED is = BD; ∴ the whole line AD is = the sum of the two lines BD and DC. Q. E. D.

30. If ABC be a triangle, whose vertical angle BAC *is bisected by* AD, and exterior vertical angle FAC *bisected by* AE; *it is* required to prove that the rectangle

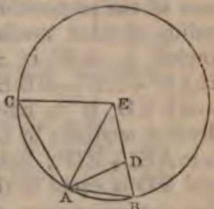
contained by BE·DC is = the rectangle contained by BD·EC.



Through C draw CG ‖ AE; then ∵ CG is ‖ AE, and AC meets them, the ∠ACG is = the ∠CAE, (Prop. 17); and ∵ CG is ‖ AE, and FG falls upon them, the exterior ∠FAE is = the interior ∠AGC; but the ∠CAE is = the ∠FAE, ∴ the ∠AGC is = the ∠ACG, and hence the side AG is = the side AC. Again, since AE is ‖ GC, a side of the △BGC, BA : AG = BE : EC, (Prop. 61); but AC is = AG, ∴ BA : AC = BE : EC; and (Prop. 60), BA : AC = BD : DC, ∴ by equality of ratios, BE : EC = BD : DC; wherefore the rectangle contained by BE·DC, is = BD·EC. Q. E. D.

Cor. If the exterior vertical angle of a triangle be bisected by a straight line, which meets the base produced, it will divide the base produced, in the same ratio as the sides of the triangle.

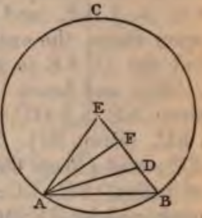
For it was proved, that BA : AC = BE : EC.



31. Let AB be a side of a decagon inscribed in the circle ABC, then the arc AB is a tenth part of the whole circumference; and ∴ the ∠AEB is a tenth part of four r^t ∠s, (Prop. 1, cor. 3), or a fifth part of two r^t∠s, ∴ the remaining ∠s EAB and EBA, of the △EAB, are together four-fifths of two r^t∠s, and they are equal, ∵ EA is = EB, hence each of them is two-fifths of two r^t∠s, and ∴ double of the ∠AEB. Bisect the ∠EAB by the |AD, then each of the ∠s EAD, DAB, is = the ∠AEB; and since the ∠s DEA, DAE, are equal, the sides ED and AD are also equal. Again, since the ∠AEB is = the ∠DAB, and the ∠ at B is common to the two △s EAB, ABD, these △s are equiangular, (Prop. 19, cor. 1), and hence EB : BA = BA : BD, (Prop. 61); and since AB is = ED, each being = AD, EB : ED = ED : BD, ∴ the rectangle EB·BD is = ED^2, and (Prop. 77, schol.) the line EB is divided medially in D, and ED is its greater segment, and it was proved = AB; ∴ AB is = the greater segment of the radius divided medially.

Again, let AC be a side of a regular hexagon, then the arc AC is a sixth part of the whole circumference; ∴ the ∠AEC is a sixth part of four r^t∠s, or a third part of two r^t∠s, hence the remaining ∠s EAC and ECA are together

= two-thirds of two $r^t\angle s$, and they are = one another, $\because$ EA is = EC; wherefore each of the $\angle s$ EAC and ECA is = a third of two $r^t\angle s$, and the $\triangle$AEC is $\therefore$ equiangular, and consequently equilateral, and AC, the side of a regular hexagon, is therefore equal to the radius of the circle in which it is inscribed.



Lastly, let AB be a side of a regular pentagon, inscribed in the circle ABC, and join EA, EB; draw AF $\perp$ to EB, make FD=FE, and join AD; then since EF is = FD, and AF common to the two $\triangle s$ AFE, AFD, and the contained $\angle s$ at F are also equal, the base AE is = the base AD, (Prop. 5), and the $\angle$AED to the $\angle$ADE. Again, since AB is a side of a pentagon, the arc AB is a fifth part of the whole circumference, and $\therefore$ the $\angle$AEB is a fifth of four $r^t\angle s$, or two-fifths of two $r^t\angle s$; wherefore the $\angle$ADE is also two-fifths of two $r^t\angle s$, and hence the $\angle$EAD is a fifth of two $r^t\angle s$; and $\therefore$ by the first part of the proposition, ED is the greater segment of the radius divided medially. Now, since AEB is an acute angled $\triangle$, AB^2 is $= AE^2+EB^2-2BE\cdot EF$ (Prop. 41) $=AE^2+EB^2-BE\cdot ED$, $\because$ ED is double of EF; but $EB^2-BE\cdot ED$ is $= EB\cdot BD$, (Prop. 35), and since EB is divided medially in D, $EB\cdot BD$ is $= ED^2$, wherefore AB^2 is $=AE^2+ED^2$; but AE is the side of a regular hexagon, and ED is equal to the side of a regular decagon; therefore the side square of a regular pentagon is equal to the side square of a regular hexagon, together with the side square of a regular decagon. Q. E. D.

Cor. 1. If each of the angles at the base of an isosceles triangle be double of the vertical angle, the base is the greater segment of the side divided medially.

Cor. 2. When a line is divided medially, the rectangle contained by the whole line and the less segment is equal to the square of the greater.

GENERAL EXERCISES IN ALGEBRA.

1st.

$$\begin{array}{r}
3a^2+4ab+b^2\\
4a^2-4ab+b^2\\
-a^2+4ab+4b^2\\
7a^2\qquad\quad-3b^2\\
5a^2+10ab+5b^2\\
\hline
18a^2+14ab+8b^2
\end{array}$$

2nd.

$$\begin{array}{r}
4a^2+16ac+16c^2\\
a^2+7ac+12c^2\\
6a^2\qquad\quad+4c^2\\
144a^2+13ac+c^2\\
11a^2+22ac+4c^2\\
3a^2+12ac+12c^2\\
\hline
169a^2+70ac+49c^2
\end{array}$$

3d.

$$\begin{array}{r}
14a+\sqrt{ac}-5c\\
13a-5\sqrt{ac}+8c\\
7a-14\sqrt{ac}\qquad\\
5a\qquad\qquad+3c\\
5a-6\sqrt{ac}+7c\\
13a+14\sqrt{ac}+3c\\
-6a+4\sqrt{ac}+4c\\
\hline
51a-6\sqrt{ac}+20c
\end{array}$$

4th.

$$\begin{array}{r}
4a+7b-3c\qquad\\
a+4ab-c\qquad\\
5a-3ab+4c\qquad\\
3a-ab+c+3d\\
4a\qquad\quad+7c-4d\\
4a-2b-c+5d\\
\hline
21a+5b+7c+4d
\end{array}$$

5th.

$$\begin{array}{r}
15a^2+12ab+13b^2\\
7a^2\qquad\quad+11b^2\\
\hline
8a^2+12ab+2b^2
\end{array}$$

6th.

$$\begin{array}{r}
17ac+5a^2+6c^2\qquad\\
17ac-5a^2+3c^2-4d\\
\hline
10a^2+3c^2+4d
\end{array}$$

7th.

Minuends.

$$\begin{array}{r}
3a^2+2ax+2x^2\\
5a^2+7ax-x^2\\
\hline
8a^2+9ax+x^2\\
4a^2+5ax-3x^2\\
\hline
4a^2+4ax+4x^2
\end{array}$$

($4a^2+5ax-3x^2$ sum of subtrahends.)

Subtrahends.

$$\begin{array}{r}
7a^2-7ax+x^2\\
-3a^2+12ax-4x^2\\
\hline
4a^2+5ax-3x^2
\end{array}$$

NOTE. The above might have been done by writing all the four quantities under each other, like an account in addition, observing to change the sign of every quantity in the subtrahend.

8th.

$$\begin{array}{r}
a^2+b^2+c^2+2ab+2ac+2bc\\
-a^2-b^2-c^2+2ab+2ac-2bc\\
\hline
2a^2+2b^2+2c^2\qquad\qquad\qquad+4bc
\end{array}$$

9th.

$$\begin{array}{rcl}
(a+b)\times c &=& ac+bc\\
(a+b)\times -d &=& -ad-bd\\
\hline
\therefore (a+b)(c-d) &=& ac+bc-ad-bd
\end{array}$$

10*th.*

$$
\begin{array}{llllllllll}
a & -b & +c & -d & \text{multiplicand} \\
a & +b & -c & -d & \text{multiplier} \\
\hline
a^2 & -ab & +ac & -ad \\
 & +ab & & & -b^2 & +bc & -bd \\
 & & -ac & & & +bc & & -c^2 & +cd \\
 & & & -ad & & & +bd & & -cd & +d^2 \\
\hline
a^2 & & & -2ad & -b^2 & +2bc & & -c^2 & & +d^2=
\end{array}
$$

$$a^2-b^2-c^2+d^2-2ad+2bc$$

11*th.*

$$
\begin{array}{llll}
x & -10 \\
x & +1 \\
\hline
x^2 & -10x \\
 & +x & -10 \\
\hline
x^2 & -9x & -10 \\
x & +4 \\
\hline
x^3 & -9x^2 & -10x \\
 & +4x^2 & -36x & -40 \\
\hline
x^3 & -5x^2 & -46x & -40
\end{array}
$$

12*th.*

$$
\begin{array}{lllll}
a^2 & +ac & +c^2 \\
a^2 & -ac & +c^2 \\
\hline
a^4 & +a^3c & +a^2c^2 \\
 & -a^3c & -a^2c^2 & -ac^3 \\
 & & a^2c^2 & +ac^3 & +c^4 \\
\hline
a^4 & & +a^2c^2 & & +c^4
\end{array}
$$

13*th.*

$$
\begin{array}{rlllll}
(a+1)(a+2)= & a^2 & +3a & +2 \\
(a+3)(a+4)= & a^2 & +7a & +12 \\
\cline{2-4}
 & a^4 & +3a^3 & +2a^2 \\
 & & 7a^3 & +12a^2 & +14a \\
 & & & 12+a^2 & +36a & +24 \\
\cline{2-6}
 & a^4 & +10a^3 & +35a^2 & +50a & +24
\end{array}
$$

14*th.*

$(a^2+ax+x^2)(a^2-ax+x^2)=a^4+a^2x^2+x^4$ See Ex. 12th and $(a-x)(a+x)=a^2-x^2$

$$
\begin{array}{llll}
a^4 & +a^2x^2 & +x^4 \\
a^2 & -x^2 \\
\hline
a^6 & +a^4x^2 & +a^2x^4 \\
 & -a^4x^2 & -a^2x^4 & -x^6 \\
\hline
a^6 & & & -x^6
\end{array}
$$

15*th.*

$$\begin{array}{l}
x^2+10\,xy+\ 7 \\
x^2-\ 6\,xy+\ 4 \\
\hline
x^4+10x^3y+\ 7x^2 \\
\quad -\ 6x^3y-60x^2y^2-42xy \\
\qquad\qquad\quad 4x^2 \quad +40xy+28 \\
\hline
x^4+\ 4x^3y-60x^2y^2 11+x^2-2xy+28
\end{array}$$

16*th.*

$$\begin{array}{l}
3x)\underline{6x^4-3x^3+12x^2+14x} \\
\quad\ 2x^3-\ x^2+\ 4x\ +\ 4\tfrac{2}{3}
\end{array}$$

17*th.*

$$\begin{array}{l}
x-3)x^4-\quad 81(x^3+3x^2+9x+27 \\
\qquad x^4-3x^3 \\
\hline
\qquad\quad 3x^3-81 \\
\qquad\quad 3x^3-\ 9x^2 \\
\hline
\qquad\qquad\quad 9x^2-81 \\
\qquad\qquad\quad 9x^2-27x \\
\hline
\qquad\qquad\qquad\quad 27x-81 \\
\qquad\qquad\qquad\quad 27x-81 \\
\hline
\end{array}$$

18*th.*

$$\begin{array}{l}
a+b)a^3+\quad b^3(a^2-ab+b^2 \\
\qquad a^3+a^2b \\
\hline
\qquad\ -a^2b+\ b^3 \\
\qquad\ -a^2b-ab^2 \\
\hline
\qquad\qquad\quad ab^2+b^3 \\
\qquad\qquad\quad ab^2+b^3 \\
\hline
\end{array}$$

19*th.*

$$\begin{array}{l}
x+y)x^5+\quad y^5(x^4-x^3y+x^2y^2-xy^3+y^4 \\
\qquad x^5+x^4y \\
\hline
\qquad\ -x^4y+\quad y^5 \\
\qquad\ -x^4y-x^3y^2 \\
\hline
\qquad\qquad\quad x^3y^2+\quad y^5 \\
\qquad\qquad\quad x^3y^2+x^2y^3 \\
\hline
\qquad\qquad\qquad\ -x^2y^3+\ y^5 \\
\qquad\qquad\qquad\ -x^2y^3-xy^4 \\
\hline
\qquad\qquad\qquad\qquad\quad xy^4+y^5 \\
\qquad\qquad\qquad\qquad\quad xy^4+y^5
\end{array}$$

20*th.*

$$\begin{array}{l}
x^2+y^3)x^5+x^2y^2+x^3y^3+y^5(x^3+y^2 \\
\qquad\quad x^5 \qquad\qquad +x^3y^3 \\
\hline
\qquad\qquad x^2y^2 \qquad\qquad +y^5 \\
\qquad\qquad x^2y^2 \qquad\qquad +y^5 \\
\hline
\end{array}$$

21*st*.

$$\begin{array}{l}
x-1)x^5-px^4+qx^3-qx^2+px-1(x^4-(p-1)x^3+(q-p+1)x^2- \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (p-1)x+1 \\
\quad \underline{x^5-x^4} \\
\qquad -(p-1)x^4+qx^3 \\
\qquad \underline{-(p-1)x^4+(p-1)x^3} \\
\qquad\qquad\qquad (q-p+1)x^3-qx^2 \\
\qquad\qquad\qquad \underline{(q-p+1)x^3-(q-p+1)x^2} \\
\qquad\qquad\qquad\qquad\qquad -(p-1)x^2+px \\
\qquad\qquad\qquad\qquad\qquad \underline{-(p-1)x^2+(p-1)x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x-1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x-1}
\end{array}$$

22*d*.

$$\begin{array}{l}
\frac{x^2}{2}-x+3\Big)\frac{3x^5}{4}-4x^4+\frac{77x^3}{8}-\frac{43x^2}{4}-\frac{33x}{4}+27\Big(\frac{3x^3}{2}-5x^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\frac{x}{4}+9 \\
\qquad \underline{\frac{3x^5}{4}-\frac{3x^4}{2}+\frac{9x^3}{2}} \\
\qquad\quad -\frac{5x^4}{2}+\frac{41x^3}{8}-\frac{43x^2}{4} \\
\qquad\quad \underline{-\frac{5x^4}{2}+5x^3-15x^2} \\
\qquad\qquad\qquad \frac{x^3}{8}+\frac{17x^2}{4}-\frac{33x}{4} \\
\qquad\qquad\qquad \underline{\frac{x^3}{8}-\frac{x^2}{4}+\frac{3x}{4}} \\
\qquad\qquad\qquad\qquad \frac{9x^2}{2}-9x+27 \\
\qquad\qquad\qquad\qquad \underline{\frac{9x^2}{2}-9x+27}
\end{array}$$

23*d*.

Since the multiplicand is the same in all the three sums here given, the sum of the products may be obtained by multiplying it by the sum of the multipliers, which is $a+b+c$.

$$\begin{array}{l}
a+b+c \\
\underline{a+b+c} \\
a^2+ab+ac \\
\qquad ab \quad +b^2+bc \\
\qquad\quad \underline{ac \quad +bc+c^2} \\
a^2+2ab+2ac+b^2+2bc+c^2 \\
\text{or } a^2+b^2+c^2+2ab+2ac+2bc
\end{array}$$

Remark. From the above it is evident, that the square of the *sum of three* quantities, is equal to the sum of the squares of these quantities, together with twice the product of every pair of the quantities. If some of the quantities had been minus, the rule would still hold, only when the quantities multiplied together had unlike signs, the sign of the product would be minus. It is universally true, that the square of the sum or difference of any number of quantities is equal to the sum of the squares of the quantities, together with twice the product of every pair of the quantities taken, with their proper signs, according to the rules of multiplication.

24*th*.

$$\begin{array}{rlll}
(a+x)(a+x)= & a^2+2ax & +x^2. & \text{Alg. art. 29.}\\
(a-x)(a-x)= & a^2-2ax & +x^2. & \text{Alg. art. 30.}\\
(a+x)(a-x)= & a^2 & -x^2. & \text{Alg. art. 31.}\\
\hline
\text{Sum } = & 3a^2 & +x^2 &
\end{array}$$

25*th*.

$$\begin{array}{rlll}
(a+x)(a+x)= & a^2+2ax & +x^2. & \text{Alg. art. 29.}\\
(a-x)(a-x)= & a^2-2ax & +x^2. & \text{Alg. art. 30.}\\
\text{Sum } = & 2a^2 & +2x^2. & \\
\text{Difference } = & \quad 4ax. & &
\end{array}$$

Remark. The above proves the following theorems:—The square of the sum, together with the square of the difference of two quantities, is equal to twice the sum of their squares; and the square of the sum, diminished by the square of the difference of two quantities, is equal to four times their product.

26*th*.

By the remark on question 23d, it is plain, that

$$\begin{array}{rl}
 & (a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc,\\
\text{and } & (a+b-c)^2=a^2+b^2+c^2+2ab-2ac-2bc;\\
\therefore \text{ the sum } & =2a^2+2b^2+2c^2+4ab,\\
\text{and the difference } = & \qquad\qquad\qquad\qquad 4ac+4bc.
\end{array}$$

27*th*.

$$\begin{array}{llllllllll}
a+b+c+d \\
a+b+c-d \\
\hline
a^2 & +ab & +ac & +ad \\
 & ab & & & +b^2 & +bc & +bd \\
 & & ac & & & +bc & & +c^2 & +cd \\
 & & & -ad & & & -bd & & -cd & -d^2 \\
\hline
a^2 & +2ab & +2ac & & +b^2 & +2bc & & +c^2 & & -d^2
\end{array}$$

$$\text{or } a^2+b^2+c^2-d^2+2ab+2ac+2bc \quad \text{(A)}$$

$$\begin{array}{llllllllll}
a+b-c+d \\
a-b+c+d \\
\hline
a^2 & +ab & -ac & +ad \\
 & -ab & & & -b^2 & +bc & -bd \\
 & & +ac & & & +bc & & -c^2 & +cd \\
 & & & ad & & & +bd & & -cd & +d^2 \\
\hline
a^2 & & & +2ad & -b^2 & +2bc & & -c^2 & & +d^2
\end{array}$$

Or $a^2-b^2-c^2+d^2+2ad+2bc$ (B)
Sum $2a^2+2ab+2ac+2ad+4bc=(A)+(B)$
Diff. $2b^2+2c^2-2d^2+2ab+2ac-2ad=(A)-(B)$

28*th*.

By the remark on question 23d, we have

$$(a+b+2c)^2=a^2+b^2+4c^2+2ab+4ac+4bc,$$

and $(a-b-2c)^2=a^2+b^2+4c^2-2ab-4ac+4bc.$

Hence the excess $=$ $4ab+8ac.$

29*th*.

$$\begin{array}{l}
a^{\frac{1}{2}}-x^{\frac{1}{2}})a^{\frac{5}{2}}-x^{\frac{5}{2}}(a+a^{\frac{1}{2}}x^{\frac{1}{2}}+x\\
\qquad\qquad \underline{a^{\frac{5}{2}}-ax^{\frac{1}{2}}}\\
\qquad\qquad\qquad ax^{\frac{1}{2}}-x^{\frac{5}{2}}\\
\qquad\qquad\qquad \underline{ax^{\frac{1}{2}}-a^{\frac{1}{2}}x}\\
\qquad\qquad\qquad\qquad a^{\frac{1}{2}}x-x^{\frac{5}{2}}\\
\qquad\qquad\qquad\qquad \underline{a^{\frac{1}{2}}x-x^{\frac{5}{2}}}
\end{array}$$

30*th*.

$$\begin{array}{l}
a^{\frac{1}{4}}-b^{\frac{1}{4}})a-b(a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}\\
\qquad\quad \underline{a-a^{\frac{3}{4}}b^{\frac{1}{4}}}\\
\qquad\qquad a^{\frac{3}{4}}b^{\frac{1}{4}}-b\\
\qquad\qquad \underline{a^{\frac{3}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}}b^{\frac{1}{2}}}\\
\qquad\qquad\qquad a^{\frac{1}{2}}b^{\frac{1}{2}}-b\\
\qquad\qquad\qquad \underline{a^{\frac{1}{2}}b^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{3}{4}}}\\
\qquad\qquad\qquad\qquad a^{\frac{1}{4}}b^{\frac{3}{4}}-b\\
\qquad\qquad\qquad\qquad \underline{a^{\frac{1}{4}}b^{\frac{3}{4}}-b}
\end{array}$$

31*st*.

$$\begin{array}{r|r|r|l}
a+4b & a^2+ab-12b^2 & a^2-5ab+6b^2 & \text{1. 1st quot.}\\
\text{2d quot.} & \underline{a^2-3ab\qquad} & \underline{a^2+ab-12b^2} & \\
 & 4ab-12b^2 & -6b)\underline{-6ab+18b^2} & \\
 & 4ab-12b^2 & a-3b & \text{G. C. M.}
\end{array}$$

32*d*.

$$\begin{array}{r|l|l|l}
3a-b & 6a^2+7ab-3b^2 & 6a^2+11ab+3b^2 & \text{1. 1st quot.}\\
\text{2d quot.} & \underline{6a^2+9ab\qquad} & \underline{6a^2+7ab-3b^2} & \\
 & -2ab-3b^2 & 2b\ )\underline{4ab+6b^4} & \\
 & -2ab-3b^2 & \qquad 2a+3b & \text{G. C. M.}
\end{array}$$

33*d*.

$x^3-a^3=(x^2+xa+a^2)(x-a)$, and $x^2-a^2=(x+a)(x-a)$; $\therefore$ (Art. 49) the least common multiple is $(x^2+ax+a^2)(x-a)(x+a)=(x^3-a^3)\times(x+a)=x^4+x^3a-xa^3-a^4$.

34*th*.

By Art. 31, the expressions which are given can be resolved into the following: $(a-1)$, $(a+1)(a-1)$, $(a-2)$, $(a+2)(a-2)$; $\therefore$ by Art. 49, the least common multiple is $(a-1)(a+1)(a-2)(a+2)=(a^2-1)(a^2-4)=a^4-5a^2+4$.

35*th*.

By Art. 31, the second expression is divisible by the first; the third has no divisor; $\therefore$ the least common multiple is $(4a^2-1)(4a^2+1)=16a^4-1$.

36*th*.

$$\frac{ab+b^2\div b}{3b^2-cb\div b}=\frac{a+b}{3b-c}.$$

37*th*.

$$\frac{5a^2+5ab\div(a+b)}{a^2-b^2\div(a+b)}=\frac{5a}{a-b}.$$

38*th*.

$$\frac{14a^2-7ax\div(2a-x)}{10ac-5cx\div(2a-x)}=\frac{7a}{5c}.$$

39*th*.

$$\frac{ac+ax+bx+bc\div(a+b)}{af+2ay+2by+bf\div(a+b)}=\frac{c+x}{f+2y}.$$

40*th*.

First find the greatest common measure of the numerator and denominator as follows:

$$\begin{array}{r|l|r|l}
-1 & a^2-b^2-c^2-2bc & a^2+b^2+c^2+2ab+2ac+2bc & 1. \\
\text{2d quot.} & \underline{-b^2-c^2-ab-ac-2bc} & \underline{a^2-b^2-c^2 \qquad\qquad -2bc} & \text{1st quot.} \\
 & \underline{a)a^2+ab+ac} & \underline{2)2b^2+2c^2+2ab+2ac+4bc} & \\
 & \quad a+b+c & b^2+c^2+ab+ac+2bc & b+c \\
 & & \underline{b^2 \qquad\quad ab \qquad +bc} & \text{3d quot.} \\
 & & c^2 \qquad +ac+bc & \\
 & & \underline{c^2 \qquad +ac+bc} &
\end{array}$$

Hence $a+b+c$ is the greatest common measure sought, *and might have* been discovered by inspection from the remark on Question 23d; for the numerator is evidently the square of $(a+b+c)$, and is consequently divisible only by itself, and $a^2-b^2-c^2-2bc$, the denominator, must hence either be divisible by $(a+b+c)$, or the fraction would be in its lowest terms already.

$$\frac{a^2+b^2+c^2+2ab+2ac+2bc \div (a+b+c)}{a^2-b^2-c^2-2bc \qquad \div (a+b+c)} = \frac{a+b+c}{a-b-c}.$$

41*st.*

$$1+\frac{a^2+b^2-c^2}{2ab} = \frac{(2ab+a^2+b^2)-c^2}{2ab} = (\text{Art. } 29).$$

$$\frac{(a+b)^2-c^2}{2ab}, \text{ by (Art. 31)}, = \frac{(a+b+c)(a+b-c)}{2ab}.$$

42*d.*

$$1-\frac{a^2+b^2-c^2}{2ab} = \frac{(2ab-a^2-b^2)+c^2}{2ab} = (\text{Art. } 30).$$

$$\frac{c^2-(a-b)^2}{2ab}, \text{ by (Art. 31)}, = \frac{(a+c-b)(b+c-a)}{2ab}.$$

43*d.*

$$\frac{c^2}{4}\left\{b^2-\left(\frac{b^2+c^2-a^2}{2c}\right)^2\right\} = \frac{\cancel{c}^2}{4}\left\{\frac{4b^2c^2-(b^2+c^2-a^2)^2}{4\cancel{c}^2}\right\}$$

$=\frac{4b^2c^2-(b^2+c^2-a^2)^2}{16}$, this expression, since $4b^2c^2$ is the square of $2bc$, by Art. 31, becomes

$$\frac{\{(2bc+b^2+c^2)-a^2\}\ \{(2bc-b^2-c^2)+a^2\}}{16} =$$

$$\tfrac{1}{16}\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}, \text{ by Art. 31}, =$$
$$\tfrac{1}{16}\{(a+b+c)(b+c-a)(a+b-c)(a+c-b)\}, \text{ or}$$
$$\frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{16}.$$

44*th.*

Since $S=\frac{a+b+c}{2}$; $\therefore 2S=a+b+c$.

Subtracting $2a$ from both sides, gives $2S-2a=b+c-a$: $\therefore 2(S-a)=b+c-a$.

Again, by subtracting first $2b$ and then $2c$ from both sides, we obtain,

$$2(S-b)=a+c-b, \text{ and}$$
$$2(S-c)=a+b-c.$$

Substituting these values in the above expression, it becomes

$$\frac{2S\times2(S-a)\times2(S-b)\times2(S-c)}{16}=$$

$$\frac{16\{S(S-a)(S-b)(S-c)\}}{16}=$$

$$S(S-a)(S-b)(S-c).$$

45*th*.

$\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}$, for mult. both sides by a^2b^2;

$\therefore a^3+b^3>ab^2+a^2b$, or $ab(a+b)$,

hence $a^2-ab+b^2>ab$, by dividing both by $(a+b)$.

$a^2-2ab+b^2>0$, by subtracting ab from both.

But $a^2-2ab+b^2=(a-b)^2$, (Art. 29), and every square is greater than 0, except the quantity itself be nothing, which can only be in the present case, by a being equal to b.

Cor. By transposing $2ab$ in the last expression, we have $a^2+b^2>2ab$; or *the sum of the squares of any two quantities, is always greater than twice their product, except these quantities be equal.*

Another proof.

Since the square of any quantity, except it be $=0$, is greater than 0, $(a-b)^2>0$,

or $a^2-2ab+b^2>0$;

$\therefore a^2-ab+b^2>ab$, by adding ab to both sides,

$a^3+b^3>a^2b+ab^2$, by mult. by $(a+b)$.

hence $\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{b}+\frac{1}{a}$, by dividing by a^2b^2.

46*th*.

$a^3+1>a^2+a$, for dividing both sides by $(a+1)$, gives $a^2-a+1>a$;

$\therefore a^2-2a+1>0$, by sub. a from both,

or $(a-1)^2>0$, which it must be, except a be equal to 1.

47*th*.

$(x+a)^3-x^3<7ax^2$, if $a<x$.

or $3ax^2+3a^2x+a^3<7ax^2$, by expanding;

$\therefore 3+3\frac{a}{x}+\frac{a^2}{x^2}<7$, by dividing by ax^2.

hence $3\frac{a}{x}+\frac{a^2}{x^2}<4$, by subt. 3 from both sides.

Now since $a<x$, *it* is evident that $\frac{a}{x}$ is less than 1, and

that therefore $3\frac{a}{x}\angle 3$; and if $\frac{a}{x}$ be less than 1, much more is $\frac{a^2}{x^2}\angle 1$; hence when $a\angle x$, $3\frac{a}{x}+\frac{a^2}{x^2}\angle 4$, and consequently $(x+a)^3-x^3\angle 7xa^2$, which was to be shown.

48th.

$$\begin{array}{ll}
x-\frac{c^2}{x} & \\
\frac{x}{c}+\frac{c}{x} & \\
\hline
\frac{x^2}{c}-c & =\left(x-\frac{c^2}{x}\right)\times\frac{x}{c} \\
\qquad +c-\frac{c^3}{x^2} & =\left(x-\frac{c^2}{x}\right)\times\frac{c}{x} \\
\hline
\frac{x^2}{c}\qquad -\frac{c^3}{x^2} & =\left(x-\frac{c^2}{x}\right)\left(\frac{x}{c}+\frac{c}{x}\right).
\end{array}$$

49th.

$$\begin{array}{l}
\frac{a^2}{x^3}+\frac{2c^3d^4}{x^5}-\frac{7c^2}{2a^4x^3} \\
\frac{a^2}{x^3}-\frac{2c^3d^4}{x^5}+\frac{7c^2}{2a^4x^3} \\
\hline
\frac{a^4}{x^6}+\frac{2a^2c^3d^4}{x^8}-\frac{7c^2}{2a^2x^6} \\
\qquad -\frac{2a^2c^3d^4}{x^8}-\frac{4c^6d^8}{x^{10}}+\frac{14c^5d^4}{2a^4x^8} \\
\qquad\qquad\qquad +\frac{7c^2}{2a^2x^6}+\frac{14c^5d^4}{2a^4x^8}-\frac{49c^4}{4a^8x^6} \\
\hline
\frac{a^4}{x^6}\qquad\qquad -\frac{4c^6d^8}{x^{10}}+\frac{14c^5d^4}{a^4x^8}-\frac{49c^4}{4a^8x^6}
\end{array}$$

50th.

$\frac{a^2-9a+20}{a^2-6a}\times\frac{a^2-13a+42}{a^2-5a}$, resolved into factors,

becomes $\frac{(a-5)(a-4)}{a(a-6)}\times\frac{(a-7)(a-6)}{a(a-5)}$

$$=\frac{(a^2-11+28)(a-5)(a-6)}{a^2(a-6)(a-5)}=\frac{a^2-11a+28}{a^2}.$$

Since $(x+a)(x+b)=x^2+(a+b)x+ab$, if *two binomial factors*, whose first terms are the same, be *multiplied together*, the first term of the product is the square of the common term, the second term is the sum of the other terms into the common one, and the third term is the product of the other terms; hence an expression of the form x^2+bx+c, can always be resolved into two binomial factors, provided two quantities can be found whose sum is b and whose product is c, their signs being attended to in both cases; then each of those quantities, with its proper sign, being successively annexed to the common letter, will give the two factors required. By this principle each of the numerators in the preceding example is resolved into factors.

51*st.*

$$\frac{a}{b}-\frac{c}{d}\Big)\frac{a^2}{bc}-\frac{2a}{d}+\frac{ac}{be}+\frac{bc}{d^2}-\frac{c^2}{de}\Big(\frac{a}{c}-\frac{b}{d}+\frac{c}{e}$$

$$\frac{a^2}{bc}-\frac{a}{d}$$

$$-\frac{a}{d}+\frac{ac}{be}+\frac{bc}{d^2}$$

$$-\frac{a}{d}\qquad+\frac{bc}{d^2}$$

$$\frac{ac}{be}-\frac{c^2}{de}$$

$$\frac{ac}{be}-\frac{c^2}{de}.$$

52*d.*

$$\frac{a^2-x^2}{a+2x}\div\frac{a-x}{3a+6x}=\frac{a^2-x^2}{a+2x}\times\frac{3(a+2x)}{a-x}=$$

$$\frac{(a+x)(a-x)\times 3(a+2x)}{(a+2x)(a-x)}=3(a+x).$$

53*d.*

$$\left(\frac{x}{a}-\frac{a}{x}\right)\times\left(\frac{m}{n}+\frac{n}{m}\right)=\frac{mx}{an}-\frac{am}{nx}+\frac{nx}{am}-\frac{an}{mx}$$

$$\left(\frac{x}{a}+\frac{a}{x}\right)\times\left(\frac{m}{n}-\frac{n}{m}\right)=\frac{mx}{an}+\frac{am}{nx}-\frac{nx}{am}-\frac{an}{mx}$$

$$\text{Sum,}\qquad=\frac{2mx}{an}-\frac{2an}{mx}.$$

$$\text{Difference,}=\frac{2nx}{am}-\frac{2am}{nx}.$$

54*th.*

$\frac{7a-10}{5}-\frac{3a-7}{6}-\frac{27a-30}{30}$, reduced to a common denominator, become

$$\frac{42a-60}{30}-\frac{15a-35}{30}-\frac{27a-30}{30}=$$
$$\frac{42a-60-15a+35-27a+30}{30}=\tfrac{5}{30}=\tfrac{1}{6}.$$

55*th.*

$$\begin{array}{l}
x^2-x+1 \\
x^2-x+1 \\
\hline
x^4-x^3+x^2 \\
\quad -x^3+x^2-x \\
\qquad +x^2-x+1 \\
\hline
x^4-2x^3+3x^2-2x+1
\end{array}$$

56*th.*

$$\begin{array}{l}
1+2x+\ 3x^2 \\
1+2x+\ 3x^2 \\
\hline
1+2x+\ 3x^2 \\
\quad 2x+\ 4x^2+\ 6x^3 \\
\qquad +\ 3x^2+\ 6x^3+\ 9x^4 \\
\hline
1+4x+10x^2+12x^3+\ 9x^4 \\
1+2x+\ 3x^2 \\
\hline
1+4x+10x^2+12x^3+\ 9x^4 \\
\quad 2x+\ 8x^2+20x^3+24x^4+18x^5 \\
\qquad\quad 3x^2+12x^3+30x^4+36x^5+27x^6 \\
\hline
1+6x+21x^2+44x^3+63x^4+54x^5+27x^6
\end{array}$$

57*th.*

By the binomial, or by multiplication, $\left(a-\frac{1}{a}\right)^3$,

is $a^3-3a^2\times\frac{1}{a}+3a\times\frac{1}{a^2}-\frac{1}{a^3}=$

$$a^3-3a+3\frac{1}{a}-\frac{1}{a^3}=a^3-\frac{1}{a^3}-3\left(a-\frac{1}{a}\right).$$

58*th.*

$$\begin{array}{l|l}
 & 4x^4-12x^3+25x^2-24x+16(2x^2-3x+4,\text{ root.} \\
 & 4x^4 \\
\hline
4x^2-3x & -12x^3+25x^2 \\
\quad -3x & -12x^3+\ 9x^2 \\
\hline
x^2-6x+4 & \qquad 16x^2-24x+16 \\
 & \qquad 16x^2-24x+16 \\
\hline
\end{array}$$

59*th*.

$$
\begin{array}{r|l}
 & a^4-2a^3+\frac{3a^2}{2}-\frac{a}{2}+\frac{1}{16}(a^2-a+\frac{1}{4}\text{ root.} \\
 & a^4 \\
\hline
2a^2-a & -2a^3+\frac{3a^2}{2} \\
-a & -2a^3+a^2 \\
\hline
2a^2-2a+\frac{1}{4} & \frac{a^2}{2}-\frac{a}{2}+\frac{1}{16} \\
 & \frac{a^2}{2}-\frac{a}{2}+\frac{1}{16} \\
\hline
\end{array}
$$

60*th*.

$$
\begin{array}{r|l}
 & \frac{a^3c^6}{b^3}-\frac{3a^2c^5}{b}+3abc^4-b^3c^3\left(\frac{ac^2}{b}-bc,\text{ root.}\right. \\
 & \frac{a^3c^6}{b^3} \\
\hline
\frac{3a^2c^4}{b^2}-3ac^3+b^2c^2 & -\frac{3a^2c^5}{b}+3abc^4-b^3c^3 \\
 & -\frac{3a^2c^5}{b}+3abc^4-b^3c^3 \\
\hline
\end{array}
$$

61*st*.

$$
\begin{array}{r|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1 \\
 & x^6 \\
\hline
3x^4-6x^3+4x^2 & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-8x^3 \\
\hline
3x^4-12x^3+12x^2 & 3x^4-12x^3+15x^2-6x+1 \\
3x^2-6x+1 & \\
\hline
3x^4-12x^3+15x^2-6x+1 & 3x^4-12x^3+15x^2-6x+1
\end{array}
$$

62*d*.

Let the less of the numbers be represented by x, then the other will be represented by $x+1$, and we have to show that the difference of their squares is equal to the sum of the numbers, or $2x+1$.

But $(x+1)^2-x^2=x^2+2x+1-x^2=2x+1$. Q. E. D.

63*d*.

Let the less number be represented by x, then the greater will be $x+a$, and their sum will be $2x+a$. Now $(x+a)^2-x^2$, (Art. 31), $=(2x+a)a$, which is their sum $2x+a$, multiplied into their difference a. Q. E. D.

64*th.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ represent the two fractions, then $\frac{a}{b}+\frac{c}{d}=1$, and therefore 1 may be substituted instead of it.

Now $\frac{a^2}{b^2}-\frac{c^2}{d^2}$, the difference of their squares, is by (Art. 31) $=\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{a}{b}-\frac{c}{d}\right)=\frac{a}{b}-\frac{c}{d}$, since $\frac{a}{b}+\frac{c}{d}=1$.

Therefore the difference of their squares is equal to the difference of the numbers. Q. E. D.

65*th.*

Lemma. Every odd number is either of the form of $4n-1$ or $4n+1$; for if in the second expression we make $n=0$, we have the number one; and if in the first and second we make $n=1$, the first gives 3 and the second 5. In the same manner, by making $n=2$, 3, &c., these expressions become 7, 9, 11, 13, &c.; hence it is evident that every odd number is included in the expression $2n\pm1$. Let then the odd number be represented by $(4n\pm1)$, then its square diminished by 1 is $(4n\mp1)^2-1=16n^2\mp8n=8(2n^2\mp n)$, which being eight times a quantity, is therefore divisible by 8. Q. E. D.

66*th.*

Every even number may be represented by $2n$, and every odd number by $2n+1$, for a number is even if it be divisible by 2 without a remainder, and odd if being divided by 2 it leave a remainder of 1. Two odd numbers may therefore be represented by $2n+1$, and $2n'+1$; let these be multiplied together, and their product divided by 2, if there is a remainder it is odd, if not it is even.

$$\begin{array}{r} 2n+1 \\ \underline{2n'+1} \\ 2\overline{)4nn'+2n'+2n+1}. \end{array}$$

The quotient is $2nn'+n'+n$, and a remainder of 1, consequently the product is odd.

Again, $\frac{2n\times2n'}{2}=2nn'$, and $\frac{2n(2n'+1)}{2}=2nn'+n$, *both divisible* by 2 without a remainder, and consequently *both even.* Q. E. D.

EXERCISES IN SIMPLE EQUATIONS.

67th.

$8x-4=13-\frac{1}{2}x$. Given equation.
$8\frac{1}{2}x=17$, by transposition;
$\therefore x=2$.

68th.

$2x+7+\frac{3x}{2}=6x-23$. Given equation.
$4x+14+3x=12x-46$, mult. by 2.
$60=5x$, by transp. and collecting.
$12=x$, by dividing by 5.

69th.

$12\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3x}{4}+5\frac{3}{8}$. Given equation.
$294+72x-144-56x=18x+129$, by mult. by 24.
$21\ =2x$, transp. and collecting.
$10\frac{1}{2}=x$.

70th.

$\frac{3x}{5}-\frac{7x}{10}+\frac{3x}{4}-\frac{7x}{8}=-15$. Given equation.
$24x-28x+30x-35x=-600$.
$9x=600$, by collecting and changing the signs;
$\therefore x=66\frac{2}{3}$.

71st.

$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=7x-734+\frac{x}{5}$. Given equation.
$30x+20x+15x=420x-44040+12x$, clearing of fractions,
$367x=44040$, collecting and changing the signs;
$\therefore x=120$.

72d.

$\frac{9x+7}{2}-\frac{6x+2}{7}=36$. Given equation.
$63x+49-12x-4=504$, mult. by 14.
$51x=459$, transp. and collecting;
$\therefore x=9$.

73d.

$\frac{5x-7}{3}-\frac{3x-2}{7}=\frac{x-5}{4}$. Given equation.

$140x-196-36x+24=21x-105$, mult. by 84,
$83x=67$, by transp. and collecting;
$\therefore x=\frac{67}{83}$.

74th.

$\frac{x+4}{2}+\frac{x+2}{3}=19-\frac{5x+7}{4}$. Given equation.
$6x+24+4x+8=228-15x-21$, mult. by 12,
$25x=175$, transp. and collecting;
$\therefore x=7$.

75th.

$ax+c=bx+d$. Given equation.
$(a-b)x=d-c$, by transposition;
$\therefore x=\frac{d-c}{a-b}$.

76th.

$ax+b^2=a^2+bx$. Given equation.
$(a-b)x=a^2-b^2$, by transposition;
$\therefore x=\frac{a^2-b^2}{a-b}=a+b$. (Art. 31.)

77th.

$(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$. Given equation.
$ab+(a+b)x+x^2-ab-ac=\frac{a^2c}{b}+x^2$,
$(a+b)x=\frac{a^2c}{b}+ac$, collecting the terms,
$b(a+b)x=a^2c+abc$, mult. by b;
$\therefore x=\frac{ac(a+b)}{b(a+b)}=\frac{ac}{b}$.

78th.

$\frac{x}{a}-1-\frac{dx}{c}=3ab$. Given equation.
$cx-ac-adx=3a^2bc$, mult. by ac.
$(c-ad)x=ac(1+3ab)$, by transp., &c.
$\therefore x=\frac{ac(1+3ab)}{c-ad}$.

79th.

$\frac{ab}{x}=bc+d+\frac{1}{x}$. Given equation.

$ab=(bc+d)x+1$, mult. by x.
$ab-1=(bc+d)x$, by transposition;
$\therefore x=\frac{ab-1}{bc+d}$. Dividing by $(bc+d)$, and changing the sides.

80*th*.

$a+\frac{m(a-x)}{3a+x}=c$. Given equation.
$3a^2+ax+am-mx=3ac+cx$, clearing of fractions,
$(a-m-c)x=3ac-3a^2-am$, by transposition;

$$\therefore x=\frac{a(3c-3a-m)}{a-m-c}$$

$$=\frac{a(m+3a-3c)}{m+c-a}\text{, by changing the signs.}$$

81*st*.

$\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{m}{nx}=p$. Given equation.

$adfn+bcfn+bden+bdfm=bdfnpx$, by clearing away the fractions; then changing the sides, and dividing by the coefficient of x, we obtain

$$x=\frac{adfn+bcfn+bden+bdfm}{bdfnp}.$$

82*d*.

	1	$3x+2y=118$,	Given equations.
	2	$x+5y=191$,	
$(2)\times 3$	3	$3x+15y=573$.	
$(3)-(1)$	4	$13y=455$.	
$(4)\div 13$	5	$y=35$, value of y.	
$(1)\times 2\frac{1}{2}$	6	$7\frac{1}{2}x+5y=295$.	
$(6)-(2)$	7	$6\frac{1}{2}x=104$.	
$(7)\div 6\frac{1}{2}$	8	$\therefore x=16$, value of x.	

83*d*.

	1	$\frac{1}{8}x+\frac{1}{9}y=42$	Given Equations.
	2	$\frac{1}{9}x+\frac{1}{8}y=43$	
$(1)\times 72$	3	$9x+8y=3024$.	
$(2)\times 72$	4	$8x+9y=3096$.	
$(3)+(4)$	5	$17x+17y=6120$.	
$(5)\div 17$	6	$x+y=360$.	
$(6)\times 8$	7	$8x+8y=2880$.	
$(3)-(7)$	8	$x=144$. value of x.	
$(4)-(7)$	9	$y=216$. value of y.	

84*th*.

	1	$x+y=20,$	Given equations.
	2	$x+z=38,$	
	3	$y+z=46,$	
(1)+(2)+(3)	4	$2x+2y+2z=104.$	
(4)÷2	5	$x+y+z=52.$	
(5)—(3)	6	$x=6$, value of x.	
(5)—(2)	7	$y=14$, value of y.	
(5)—(1)	8	$z=32$, value of z.	

85*th*.

	1	$y+\frac{1}{2}x=41,$	Given equations.
	2	$x+\frac{1}{4}z=20\frac{1}{2},$	
	3	$y+\frac{1}{5}z=34,$	
(1)—(3)	4	$\frac{1}{2}x-\frac{1}{5}z=7.$	
(4)×2	5	$x-\frac{2}{5}z=14.$	
(2)—(5)	6	$(\frac{1}{4}+\frac{2}{5})z=6\frac{1}{2}.$	
(6)×20	7	$13z=130.$	
(7)÷13	8	$z=10$, value of z.	
(8)÷5	9	$\frac{1}{5}z=2.$	
(3)—(9)	10	$y=32$, value of y.	
(8)÷4	11	$\frac{1}{4}z=2\frac{1}{2}.$	
(2)—(11)	12	$x=18$, value of x.	

86*th*.

	1	$2x+5y-7z=-288,$	Given equations.
	2	$5x-y+3z=227,$	
	3	$7x+6y+z=297,$	
(3)×3	4	$21x+18y+3z=891.$	
(3)×7	5	$49x+42y+7z=2079.$	
(4)—(2)	6	$16x+19y=664.$	
(5)+(1)	7	$51x+47y=1791.$	
(6)×3	8	$48x+57y=1992.$	
(8)—(7)	9	$-3x+10y=201.$	
(9)×16	10	$-48x+160y=3216.$	
(8)+(10)	11	$217y=5208.$	
(11)÷217	12	$\therefore y=24$, value of y.	
(12)×10	13	$10y=240.$	
(13)—9	14	$3x=39.$	
(14)÷3	15	$\therefore x=13$, value of x.	
(15)×5—(12)	16	$5x-y=41.$	
(2)—(16)	17	$3z=186.$	
(17)÷3	18	$\therefore z=62$, value of z.	

87*th*.

Let $x=$ the money in shillings,
Then $\frac{x}{4}+\frac{x}{5}=11\frac{1}{4}$, by the question.
$5x+4x=225$, mult. by 20.
$9x=225$, collecting the terms.
$\therefore x=25$ shillings, or L.1, 5s.

88*th*.

Let $x=$ the number of students,
Then $2x=$ the number of officers,
And $4x=$ the number of merchants;
Hence $x+2x+4x=266$, by the question,
$7x=266$, collecting the terms;
$\therefore x=38$, the students,
$2x=76$, the officers,
And $4x=152$, the merchants.

89*th*.

Let $x=$ the first part,
Then mx will $=$ the second part,
And nx will $=$ the third part;
Hence $x+mx+nx=a$, by the question,
Or $(1+m+n)x=a$;

$$\therefore x=\frac{a}{1+m+n}, \text{ the first part,}$$

$$mx=\frac{ma}{1+m+n}, \text{ the second part,}$$

$$\text{And } nx=\frac{na}{1+m+n}, \text{ the third part.}$$

90*th*.

Let $x=$ the pounds the youngest receives,
Then $x+20=$ the second's share,
$x+40=$ the third's share,
$x+60=$ the fourth's share,
And $x+80=$ the eldest's share.
Therefore the sum of all their shares, which is
$5x+200=1000$, by the question,
$5x\quad=800$, by transposition;
$\therefore x\quad=160$, the youngest's share.

91*st.*

Let $x=$ the days in which the second will overtake the first; then since the first started n days sooner, $x+n$ will be the days the first took to travel the whole distance; and as he travelled a miles per day, the whole distance travelled will be $a(x+n)$ miles; but since the second travelled it in x days, at the rate of b miles per day, the whole distance will also be bx, and hence

$$bx=a(x+n), \text{ each being } = \text{ the whole distance,}$$

$$\text{or } bx=ax+an,$$

$$(b-a)x=an, \text{ by transposition;}$$

$$\therefore x=\frac{an}{b-a}, \text{ the time required.}$$

NOTE. If a were greater than b, the answer would be minus, which shows that the second would never overtake the first; and also, that if they came from the other side of the point from which they were supposed to start, and past it at the distance described, they had been together $\frac{an}{a-b}$ days before the second reached that place. Again, if $b=a$, the fraction $\frac{an}{b-a}=\frac{an}{0}=$ infinity; consequently, on the supposition in the question, they will never be together, which is evident, since they would then both walk at the same rate.

92*d.*

Let $x=$ the crowns, then $17-x$ will be the shillings; also $5x$ will be the shillings paid in crowns.

$$\text{Hence } 5x+17-x=53 \text{ shillings, by the question,}$$

$$\text{Or } 4x=36, \text{ by transposition,}$$

$$\therefore x=9, \text{ the number of crowns;}$$

$$\text{And } 17-9=8, \text{ the number of shillings.}$$

93*d.*

Let $x=$ the years sought; then the father's age will be $40+x$, and the son's will be $9+x$; and by the question, twice $9+x$ is equal to $40+x$; or

$$2(9+x)=40+x,$$

$$18+2x=40+x,$$

$$\therefore x=22, \text{ by transposition.}$$

94*th.*

Let $x=$ the sheets transcribed weekly by working four hours per day, then $4:10=x:\frac{10x}{4}=2\frac{1}{2}x$ the sheets that would be transcribed by working ten hours per day; but by the question this latter number would be as much above 70 as the former was below it, consequently

$$2\tfrac{1}{2}x-70=70-x$$

$3\tfrac{1}{2}x=140$ by transposition.

$\therefore x=40$ sheets weekly.

95*th*.

Let x = the first number, and y = the second.

Then	1	$2x+5y=31$	By the question.
And	2	$7x+4y=68$	
(1)+(2)	3	$9x+9y=99$	
(3)× $\frac{1}{9}$	4	$4x+4y=44$	
(2)—(4)	5	$3x\quad=24$	
(5)÷ 3	6	$\therefore x=8$ value of x, the first.	
(4)÷ 2,	7	$2x+2y=22$	
(1)—(7)	8	$3y=9$	
(8)÷ 3	9	$y=3$ value of y, the second.	

96*th*.

Let x = the numerator, and y = the denominator.

Then	1	$\frac{x-3}{y-3}=\frac{1}{4}$	By the question.
	2	$\frac{x+5}{y+5}=\frac{1}{2}$	
(1)×4(y—3)	3	$4x-12=y-3$	
	4	$4x-y=9$ by transposition.	
(2)×2(y+5)	5	$2x+10=y+5$	
	6	$2x-y=-5$ by transposition.	
(4)— (6)	7	$2x\quad=14$	
(7)÷ 2	8	$\therefore x=7$ the numerator.	
(6)× 2	9	$4x-2y=-10$	
(4)— (9)	10	$y=19$ the denominator.	

97*th*.

Let x = the greater number, then since their sum is a, the less may be represented by $(a-x)$; hence

$x^2-(a-x)^2=b$, by the question,

$2ax-a^2=b$ expanding and collecting,

$2ax=a^2+b$ by transposition;

$\therefore x=\frac{a^2+b}{2a}$ the greater;

and $a-\frac{a^2+b}{2a}=\frac{2a^2-a^2-b}{2a}=\frac{a^2-b}{2a}$ the less.

98*th*.

Let x=*the one*, then $(a-x)$ will be the other.

$$\frac{x}{a-x}=b \text{ by the question,}$$

$$x=ab-bx \text{ mult. by } (a-x),$$

$$(b+1)x=ab \text{ by transposition;}$$

$$\therefore x=\frac{ab}{b+1} \text{ dividing by } (b+1),$$

$$a-x=a-\frac{ab}{b+1}=\frac{ab+a-ab}{b+1}=\frac{a}{b+1}.$$

99*th*.

Let x=A's money, y=B's money, and z=C's money.

Then	1	$x+\frac{3}{7}y=2190$	By the question.
	2	$y+\frac{5}{9}z=2190$	
	3	$\frac{2}{3}x+z=2190$	
(1)×21	4	$21x+9y=45990$	
(2)× 9	5	$9y+5z=19710$	
(4)—(5)	6	$21x-5z=26280$	
(3)× 3	7	$2x+3z=6570$	
(7)× 10½	8	$21x+31\frac{1}{2}z=68985$	
(8)—(6)	9	$36\frac{1}{2}z=42705$	
(9)÷ 36½	10	$\therefore z=1170$	C's money.
(3)—(10)	11	$\frac{2}{3}x=1020$	
(11)× $\frac{3}{2}$	12	$\therefore x=1530$	A's money.
(1)—(12)	13	$\frac{3}{7}y=660$	
(13)× $\frac{7}{3}$	14	$\therefore y=1540$	B's money.

100*th*.

Let the first = x, the second = y, and the third = z.

Then	1	$x+6:y+6=2:3$	By the question.
	2	$x+5:z+5=7:11$	
	3	$y-36:z-36=6:7$	
From (1)	4	$3x-2y=-6$	Mult. extremes and means and transposing.
From (2)	5	$11x-7z=-20$	
From (3)	6	$7y-6z=36$	Do.
(4)× 3½	7	$10\frac{1}{2}x-7y=-21$	
(6)+(7)	8	$10\frac{1}{2}x-6z=15$	
(8)× 7	9	$73\frac{1}{2}x-42z=105$	
(5)× 6	10	$66x-42z=-120$	
(9)—(10)	11	$7\frac{1}{2}x=225$	
(11)÷ 7½	12	$\therefore x=30$ the first.	
(12)× 3	13	$3x=90$	
(13)—(4)	14	$2y=96$	
(14)÷ 2	15	$\therefore y=48$ the second.	
(12)× 11	16	$11x=330$	
(16)—(5)	17	$7z=350$	
(17)÷ 7	18	$\therefore z=50$ the third.	

QUADRATIC EQUATIONS.

101*st.*

$x^2+6x=27$ given equation,
$x^2+6x+9=36$ completing the square,
$x+3=\pm 6$ extracting the root;
$\therefore x=-3\pm 6=3$, or -9.

102*d.*

$x^2-7x+3\frac{1}{4}=0$ given equation.
$x^2-7x=-3\frac{1}{4}$ by transposing.
$x^2-7x+12\frac{1}{4}=9$ completing the square,
$x-3\frac{1}{2}=\pm 3$ extracting the root;
$\therefore x=3\frac{1}{2}\pm 3=6\frac{1}{2}$, or $\frac{1}{2}$.

103*d.*

$x^2-5\frac{3}{4}x=18$ given equation.
$x^2-5\frac{3}{4}x+\left(\frac{23}{8}\right)^2=\frac{1681}{64}$ completing the square.
$x-\frac{23}{8}=\pm\frac{41}{8}$.
$x=\frac{23}{8}\pm\frac{41}{8}=8$, or $-2\frac{1}{4}$.

104*th.*

$3x^2-2x=65$ given equation.

Multiplying by 12 and adding the square of 2 to both sides gives $36x^2-24x+4=784$.

$6x-2=\pm 28$ extracting the root.
$6x=2\pm 28$ transposing;
$\therefore x=5$, or $-4\frac{1}{3}$.

105*th.*

$\frac{x}{x+60}=\frac{7}{3x-5}$ given equation.
$3x^2-5x=7x+420$ clearing of fractions,
$3x^2-12x=420$ by transposing,
$x^2-4x=140$ by dividing by 3,
$x^2-4x+4=144$ completing the square,
$x-2=\pm 12$ extracting the root;
$\therefore x=2\pm 12=14$, or -10.

106*th.*

$x+\frac{24}{x-1}=3x-4$ given equation.
$x^2-x+24=3x^2-7x+4$ clearing of fractions,
$2x^2-6x=20$ *by transposing and changing the sides,*

$x^2-3x=10$ dividing by 2,
$x^2-3x+\frac{9}{4}=\frac{49}{4}$ completing the square,
$x-\frac{3}{2}=\pm\frac{7}{2}$ extracting the root;
$\therefore x=\frac{3}{2}\pm\frac{7}{2}=5$, or -2.

107*th.*

$a^2(1+b^2x^2)=b(2a^2x+b)$ given equation,
$a^2+a^2b^2x^2=2a^2bx+b^2$,
$a^2b^2x^2-2a^2bx=b^2-a^2$ by transposing,
$a^2b^2x^2-2a^2bx+a^2=b^2$ completing the square,
$abx-a=\pm b$ by extracting the root,
$abx=a\pm b$ by transposition;
$\therefore x=\frac{a+b}{ab}$, or $\frac{a-b}{ab}$.

108*th.*

$\sqrt{a+x}+\sqrt{a-x}=\frac{12a}{5\sqrt{a+x}}$ given equation,
$5a+5x+5\sqrt{a^2-x^2}=12a$ mult. by $5\sqrt{a+x}$,
$5\sqrt{a^2-x^2}=7a-5x$ by transposition,
$25a^2-25x^2=49a^2-70ax+25x^2$ by squaring,
$50x^2-70ax=-24a^2$ transposing and changing the signs,
$25x^2-35ax=-12a^2$ dividing by 2,

$$\left.\begin{array}{l}2500x^2-3500ax+1225a^2=25a^2\\ 50x-35a=\pm5a\end{array}\right\}\text{Art. 81, Alg.}$$

$50x=35a\pm5a=40a$, or $30a$;
$\therefore x=\frac{4a}{5}$, or $\frac{3a}{5}$ by dividing by 50.

109*th.*

$(a-b)x^2-(a+b)x+2b=0$ given equation,
$(a-b)x^2-(a+b)x=-2b$ by transposition,
$4(a-b)^2x^2-4(a^2-b^2)x+(a+b)^2=a^2-6ab+9b^2$,
$2(a-b)x-(a+b)=\pm(a-3b)$ Alg. Art. 81.
$2(a-b)x=a+b\pm(a-3b)=2(a-b)$, or $4b$;
$\therefore x=1$, or $\frac{2b}{a-b}$.

110*th.*

$x+4+\left(\frac{x+4}{x-4}\right)^{\frac{1}{2}}=\frac{12}{x-4}$ given equation,
$\left(\frac{x+4}{x-4}\right)^{\frac{1}{2}}=\frac{12}{x-4}-x-4$ by transposition,

$\left(\frac{x-4}{x-4}\right)^{\frac{1}{2}}=\frac{28-x^2}{x-4}$ by reduction.

$\frac{(x^2-16)^{\frac{1}{2}}}{x-4}=\frac{28-x^2}{x-4}$ by multiplying numerator and denominator of the first side by $(x-4)^{\frac{1}{2}}$.

$x^2-16=784-56x^2+x^4$ clearing of fractions and squaring.

$x^4-57x^2=-800$ by transp. and changing the sides.

$x^4-57x^2+\frac{3249}{4}=\frac{49}{4}$ Alg. Art. 82.

$x^2-\frac{57}{2}=\pm\frac{7}{2}$ extracting the root.

$x^2=\frac{57}{2}\pm\frac{7}{2}=32$, or 25 by transposition.

$\therefore x=\pm4\sqrt{2}$, or ±5 by extracting the root.

111*th*.

$x^6-7x^3=8$ given equation.

$x^6-7x^3+\frac{49}{4}=\frac{81}{4}$ completing the square, Art. 82.

$x^3-3\frac{1}{2}=\pm4\frac{1}{2}$ extracting the root.

$x^3=8$ or -1 by transposition.

$\therefore x=2$, or -1 extracting the cube root.

112*th*.

	1 2	$\left.\begin{array}{r}x^2+y^2=a\\ x^2-y^2=b\end{array}\right\}$ Given equations.
(1)+(2)	3	$2x^2=a+b$
(3)÷ 2	4	$x^2=\frac{a+b}{2}$
Square root of (4)	5	$\therefore x=\pm\sqrt{\frac{a+b}{2}}$ value of x.
(1)—(2)	6	$2y^2=a-b$
(6)÷ 2	7	$y^2=\frac{a-b}{2}$
Square root of (7)	8	$y=\pm\sqrt{\frac{a-b}{2}}$ value of y.

113*th*.

	1 2	$\left.\begin{array}{r}x+y=a\\ xy=b\end{array}\right\}$ Given equations.
(1) squared	3	$x^2+2xy+y^2=a^2$
(2)× 4	4	$4xy=4b$
(3)—(4)	5	$x^2-2xy+y^2=a^2-4b$
Square root of (5)	6	$x-y=\pm\sqrt{a^2-4b}$
(1)+(6)	7	$2x=a\pm\sqrt{a^2-4b}$
(7)÷ 2	8	$\therefore x=\frac{a\pm\sqrt{a^2-4b}}{2}$ value of x.

(1)—(6)	9	$2y=a\mp\sqrt{a^2-4b}$
(9)÷ 2	10	$\therefore y=\frac{a\mp\sqrt{a^2-4b}}{2}$ value of y.

114*th.*

	1	$x^2+xy+y^2=52$ Given
	2	$xy-x^2=8$ equations.
Let $y=vx$ (1)	3	$x^2+vx^2+v^2x^2=52$
and we have (2)	4	$vx^2-x^2=8$
(3)÷(4) gives	5	$\frac{1+v+v^2}{v-1}=\frac{13}{2}$
(5)×2(v—1)	6	$2v^2+2v+2=13v-13$
(6) transposed	7	$2v^2-11v=-15$
Completing square	8	$16v^2-88v+121=1$
Square root of (8)	9	$4v-11=\pm 1$
	10	$\therefore v=3$, or $2\frac{1}{2}$
By substitution in (4)	11	$2x^2$, or $\frac{3}{2}x^2=8$
	12	$\therefore x=\pm 2$, or $\pm\frac{4}{\sqrt{3}}$ values of x.
Since $y=vx$	13	and $y=\pm 6$, or $\pm\frac{10}{\sqrt{3}}$ values of y.

115*th.*

	1	$x^2-2xy-y^2=31$ Given
	2	$\frac{1}{2}x^2+2xy-y^2=101$ equations.
Let $y=vx$, (1)	3	$x^2(1-2v-v^2)=31$
and we have (2)	4	$x^2(\frac{1}{2}+2v-v^2)=101$
(3)÷(4)	5	$\frac{1-2v-v^2}{\frac{1}{2}+2v-v^2}=\frac{31}{101}$
(5)×202($\frac{1}{2}+2v-v^2$)	6	$202-404v-202v^2=31+124v-62v^2$
(6) transp.	7	$140v^2+528v=171$, changing the sides
(7) Alg. art. 80.	8	$280v+528=\pm 612$
(8)÷280.	9	$\therefore v=\frac{\pm 612-528}{280}=\frac{3}{10}$, or $-\frac{57}{14}$.
(3) by subst. $v=\frac{3}{10}$	10	$x^2\times\frac{31}{100}=31$
(10)×$\frac{100}{31}$	11	$x^2=100$
extracting root of (11)	12	$\therefore x=\pm 10$, values of x,
since $y=vx$	13	$vx=y=\pm 10\times\frac{3}{10}=\pm 3$, values of y.

116*th.*

1	$\frac{x}{x+y}-\frac{y}{x}=\frac{x^2-y^2}{18}$ Given
2	$\frac{x}{y}-\frac{x+y}{x}=\frac{y}{x}$ equations.

Put vx for y in (2), and dividing out x,	3	$\frac{1}{v}-1-v=v$,
(3)×v,	4	$1-v-v^2=v^2$, or $2v^2+v=1$,
(4) Alg. Art. 80,	5	$4v+1=\pm 3$,
(5)÷4,	6	$\therefore v=\frac{1}{2}$, or -1; hence $y=\frac{1}{2}x$.
(1) by putting $y=\frac{1}{2}x$,	7	$\frac{x}{x+\frac{1}{2}x}-\frac{\frac{1}{2}x}{x}=\frac{x^2-\frac{1}{4}x^2}{18}$,
(7) reduced,	8	$\frac{2}{3}-\frac{1}{2}=\frac{\frac{3}{4}x^2}{18}$,
(8) ×18	9	$12-9=\frac{3}{4}x^2$,
(9)×$\frac{4}{3}$,	10	$x^2=4$, and changing the sides,
ext. root of (10)	11	$\therefore x=\pm 2$, values of x,
	13	$vx=y=\pm 2\times\frac{1}{2}=\pm 1$, values of y.

NOTE. In the last two questions, the second values of v do not give consistent values of x and y; in the 115, by substituting $-\frac{57}{14}$ for v, x^2 is equal to a minus quantity, and therefore impossible; and in 116, by putting $v=-1$, one of the terms becomes infinite.

117*th.*

Let $x=$ the number sought; then,

$\frac{x}{2}\times\frac{x}{3}=864$, by the question,

$x^2=5184$, mult. both sides by 6,

$\therefore x=72$.

118*th.*

Let $x=$ the number sought; then,

$(94+x)(94-x)=8512$, by the question,

or $8836-x^2=8512$,

$\therefore x^2=324$, by transp. and changing signs,

and hence $x=18$.

119*th.*

Let $x=$ the less of the two numbers; then, since their quotient is 9, the greater will be $9x$.

and $9x\times x=144$, by the question,

or $9x^2=144$,

$x^2=16$, dividing by 9,

$$\left.\begin{array}{r}\therefore x=4,\\ \text{and } 9x=36,\end{array}\right\}\text{ the numbers sought.}$$

120*th.*

Let $x=$ *the less of the two numbers*; then, since their *quotient is* b, *the other* will be bx.

$$x \times bx = a,\text{ by the question,}$$
$$bx^2 = a,$$
$$x^2 = \frac{a}{b},$$
$$\therefore x = \sqrt{\frac{a}{b}},\text{ the less;}$$
$$\text{and } bx = b\sqrt{\frac{a}{b}} = \sqrt{ab},\text{ the greater.}$$

121*st.*

Let $x=$ the share of the first, and $r=$ the common ratio; then $rx=$ the second's, and $r^2x=$ the third's share.

and	1	$x+rx+r^2x=210$ } By the question.
	2	$x-r^2x \quad =90$
(1)÷(2)	3	$\frac{1+r+r^2}{1-r^2}=\frac{7}{3}$
(3)×$3(1-r^2)$	4	$3r^2+3r+3=7-7r^2$
(4) transp.	5	$10r^2+3r=4$
(5) Alg. Art. 80	6	$20r=\pm13-3$;
(6)÷20	7	$\therefore r=\frac{1}{2}$
(2) by putting $\frac{1}{4}=r^2$	8	$x-\frac{1}{4}x=90$
(8)×4	9	$3x=360$
(9)÷3	10	$x=120$, the first's share,
(10)×$\frac{1}{2}$	11	$\frac{1}{2}x=60$, the second's share,
(11)×$\frac{1}{2}$	12	$\frac{1}{4}x=30$, the third's share.

122*d.*

Let $x=$ the capital required; then, by the rules for calculating interest, its interest at 4 per cent. for 5 months will be $\frac{x\times4\times5}{1200}=\frac{x}{60}$; but, by the question,

$$x\times\frac{x}{60}=117041\tfrac{2}{3},$$
$$x^2=7022500,\text{ by mult. by 60,}$$
$$\therefore x=2650,\text{ by extracting the root.}$$

123*d.*

Let $x=$ the less, then $x+8$ will $=$ the greater of the numbers;

hence $x(x+8)=240$, by the question,

$x^2+8x+16=256$, completing the square,

$x+4=\pm16$, extracting the root,

$\therefore x=12$, the less,

and $x+8=20$, the greater.

124*th*.

Let $x=$ the number sought, then by the question we have

$x^2-x=306$,

$x^2-x+\frac{1}{4}=\frac{1225}{4}$, completing the square.

$x-\frac{1}{2}=\frac{35}{2}$, extracting the root,

$\therefore x=\frac{1}{2}+\frac{35}{2}=18$.

125*th*.

Let $x=$ the number sought; then its third part multiplied by its fourth part, will be $\frac{x}{3}\times\frac{x}{4}=\frac{x^2}{12}$; and adding five times the required number, gives $\frac{x^2}{12}+5x=\frac{x^2+60x}{12}$; hence $\frac{x^2+60x}{12}-200=280-x$, by the question,

$x^2+60x-2400=3360-12x$, mult. by 12,

$x^2+72x=5760$, by transposition,

$x^2+72x+1296=7056$, completing the square,

$x+36=84$, extracting the root,

$\therefore x=84-36=48$.

126*th*.

Let $x=$ the pieces bought, then $\frac{60}{x}$ will represent the price per piece in pounds; but if he had bought three pieces more for the same money, each piece would have cost $\frac{60}{x+3}$ pounds; hence, since this quotient by the question is one pound less than the former, we have the following equation:

$\frac{60}{x+3}+1=\frac{60}{x}$,

$60x+x^2+3x=60x+180$, clearing of fractions,

$x^2+3x=180$, subt. $60x$ from both sides,

$x^2+3x+\frac{9}{4}=\frac{729}{4}$, completing the square,

$x+\frac{3}{2}=\frac{27}{2}$, extracting the root,

$x=\frac{27}{2}-\frac{3}{2}=12$, the number of pieces.

127*th*.

Let $x=$ the number of children, then after two of them die, there will be $(x-2)$, and by the question,

$\frac{46800}{x}=\frac{46800}{x-2}-1950,$

$46800x-93600=46800x-1950x^2+3900x.$

Transposing and dividing by 1950, the equation becomes

$x^2-2x=48,$

$x^2-2x+1=49,$ completing the square,

$x-1=7,$

$\therefore x=8.$

128*th.*

Let $x=$ the person's stock which was in for five months, then $(500-x)=$ the other's stock; also let $y=$ the gain per pound, in one month, consequently the first's gain would be $5xy$, and the second's would be $2(500-x)\,y$;

and $\left.\begin{array}{l} x+5xy=450 \\ 500-x+2(500-x)y=450 \end{array}\right\}$ By the question.

$y=\frac{450-x}{5x}$ from the first.

$500-x+2(500-x)\frac{450-x}{5x}=450$ since $y=\frac{450-x}{5x}$

$2500x-5x^2+450000-1900x+2x^2=2250x.$

Transposing $3x^2+1650x=450000$ and changing the signs.

$x^2+\ 550x=150000$ dividing by 3.

$x^2+\ 550x+75625=225625$

$x\ +\ 275\ =\pm475$

$\therefore x\ =\ 475-275=200$

and $500-x=500-200=300.$

129*th.*

Let $x=$ the one, then $(41-x)$ will be the other, and

$(41-x)^2+x^2=901$ by the question.

$1681-82x+2x^2=901$

$x^2-41x=-390$ transposing and dividing by 2.

$x^2-41x+\frac{1681}{4}=\frac{121}{4}$ completing the square.

$x\ -\frac{41}{2}=\pm\frac{11}{2}$ extracting the root.

$\therefore x=\frac{41}{2}\pm\frac{11}{2}=26$ or 15.

130*th.*

The amount of L.1 at 4 per cent. for 40 years, compound interest, is by Table 7=4.801021, which being multiplied by 5000 gives L.24005·105=L.24005, 2s. 1¼d.

131*st.*

L.5000 for 12 years at 4 per cent.=(By table 7) 1·601032 $\times$ *5000=L.8005·160,* and this amount divided by 3600 gives 2·223655, which by inspection of the table is found to lie between the amount for 16 years and the amount for 17 years; and on calculating for the difference, it is found to be 16 years 136 days. The same problem solved by logarithms gives the answer 16 years 138 days..

132*d*.

L.4500 at 6 per cent. in 9 years will amount (Table 7) to 1·689479 × 4500=L.7602·6555; this being divided by the amount of L.1 at 4 per cent. for 15 years, will give the sum sought; that is

$$\frac{7602.6555}{1.800944}=\text{L.}4221{\cdot}483.$$

133*d*.

This question is the same, so far as calculation is concerned, as the following:—Find the present worth of L.20,000 due ten years hence at 3 per cent.; which can be calculated by Table 8, by taking out the present value of L.1 due ten years hence at 3 per cent., and multiplying it by 20,000, thus ·744094 × 20,000=14881·98=14,882 nearly. Or, since the population increased in such manner that 100 at the beginning of a year became 103 at the end of the same, the population at the end of a year is to that at its beginning as 103 : 100; if we then put n for the population at any time, the population a year before that time would be $\frac{100n}{103}$; and since 103: 100 : : $\frac{100n}{103}$: $\frac{(100)^2n}{(103)^2}$, we may in the same manner deduce that the population ten years before that time would be $\frac{(100)^{10}n}{(103)^{10}}$, in which expression, if n be put equal to 20,000, and the value found, it will be the same as that given above; and the calculation may either be performed by natural numbers, or by logarithms.

134*th*.

From what was said on the previous question, it will appear plain that we have here to find what power of the fraction $\frac{103}{100}$ or of 1·03 will give the result ten; or to solve the equation $(\frac{103}{100})^x=10$, or $(1{\cdot}03)^x=10$. Taking the logarithms of both sides gives x Log. 1·03=1; ∴ $x=\frac{1}{\text{Log. }1{\cdot}03}=\frac{1}{{\cdot}012837}=78$ nearly.

135*th*.

In Table 10, under 6 per cent., and opposite to 40 years, the present value of L.1 at 6 per cent. is found to be L.15·046297, which being multiplied by 20, gives 300·925940=L.300, 18s. 6d.

136*th.*

In Table 9, under 4 per cent., and opposite to 12 years, the amount of L.1 for 21 years at 4 per cent. is found to be L.15·025805, the given amount, namely L.$500{\cdot}8641\dot{6}$, being divided by the tabular number, will give the annuity sought; or $\frac{500{\cdot}864166}{15{\cdot}025805}$ =33·333=L.33, 6s. 8d.

END OF KEY TO PART FIRST.

N^o^. XVII. PART II.

KEY

TO

SYSTEM OF PRACTICAL MATHEMATICS,

PART II.

CONTAINING

SOLUTIONS OF ALL THE EXERCISES IN THAT WORK.

BY THE

SCOTTISH SCHOOL-BOOK ASSOCIATION.

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PREFACE.

THIS Work has been drawn up for the convenience of Teachers, and the aid of private Students. Hence most of the solutions are given very fully, and wherever abridgments occur, the operations are so clearly indicated as to prevent the possibility of mistaking the different steps of the process. In it, therefore, the Student will always find a ready solution to any difficulty he may encounter in the SECOND PART of the Work on PRACTICAL MATHEMATICS.

Still more to enhance the value of the KEY, a *Supplement* has been added, containing demonstrations of several Rules, which the limits prescribed to the work did not admit of being there inserted; and these have been preceded by such matter as was considered necessary to render them intelligible to those unacquainted with the Differential and Integral Calculus. These illustrations are given in as concise and simple a form as possible, and it is hoped will not only appear quite intelligible to the industrious *Student*, but will even inspire him with an ardent desire to pursue still farther that sublime subject, which the immortal Newton used as a Key to unlock the secrets of nature, and unfold the doctrines of Astronomy. It contains also the method of calculating Logarithmic Tables, and the shortest way of finding the Circumference of the Circle.

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KEY TO PART II.

LOGARITHMIC ARITHMETIC.

Art. 18.

Log. of 75·825 =1·879812
Log. of 84·75 =1·928140
Log. of Product 6426·16=3·807952

Log. of ·75 $=\bar{1}$·875061
Log. of ·0625 $=\bar{2}$·795880
Log. of ·846·25 =2·927497
Log. of Product 39.6678 =1·598438

Log. of 4·825 =0·683497
Log. of .0225 $=\bar{2}$·352183
Log. of .0145 $=\bar{2}$·161368
Log. of Product .00157415$=\bar{3}$·197048

Log. of ($7\frac{1}{4}$ =7·25) =0·860338
Log. of ($2\frac{1}{5}$ =2·2) =0·342423
Log. of ($3\frac{1}{8}$ =3·125) =0·494850
Log. of ($20\frac{5}{16}$=20·3125) =1·307763
Log. of Product 1012·45 =3·005374

Log. of ·7345 $=\bar{1}$·865992
Log. of 734.5 =2·865992
Log. of 73.45 =1·865992
Log. of Product 39625.6=4·597976

Log. of ·25$=\bar{1}$·397940
Log. of ·7325$=\bar{1}$·864808
Log. of ·01725$=\bar{2}$·236789
Log. of Product ·00315893$=\bar{3}$·499537

Art. 19.

(1.) Log. of 385 =2·585461 2·585461
Log. of 29·25 =1·466126 ar. co.= 8·533874
Log. of Quot. 13·1624=1.119335 =1·119335

(2.) Log. of 463·28 =2·665844 2·665844
Log. of 47.4 =1·675778 ar. co.= 8·324222
Quotient 9.77386=0·990066 0·990066

(3.) Log. of 364 =2·561101
Log. of 54·75 =1·738384
Log. of Product =4·299485 4·299485
Log. of ·7854 =$\bar{1}$·895091 ar. co. =10·104909
Quotient 25374·3=4·404394 4·404394

(4.) Log. of 463 =2·665581
Log. of 25·25 =1·402261
Product 4·067842 4.067842
Log. of 3.1416 =0·497151 ar. co. = 9·502849
Quotient 3721·27=3·570691 3·570691

(5.) Log. of 1 =0·000000 0·000000
Log. of 3·1416 =0·497151 ar. co. = 9·502849
Quotient ·318309=$\bar{1}$·502849 $\bar{1}$·502849

(6.) Log. of 3·1416 =0·497151 0·497151
Log. of 180 =2·255273 ar. co. = 7·744727
Quotient ·017453=$\bar{2}$·241878 $\bar{2}$·241878

PROPORTION. Art. 20.

(1.) 220 yds. : 880 yds. : : 25 labourers.
40 in. : 42 in.
30 in. : 28 in.
36 days : 32 days
8 ho. : 9 ho.

Log. 220=2·342423	Log. 880=2·944483
„ 40=1·602060	„ 42=1·623249
„ 30=1·477121	„ 28=1·447158
„ 36=1·556303	„ 32=1·505150
„ 8=0·903090	„ 9=0·954243
7·880997	„ 25=1·397940
	9·872223

Subtract sum of logs. of the first terms =7·880997
The answer. Log. 98=1·991226

(2.) 12 ho. : 10 ho. : : 3 men.
150 yds. : 192 yds.
240 yds. : 300 yds.
4 days : 5 days

Log. 12=1·079181	Log. 10=1·000000
,, 150=2·176091	,, 192=2·283301
,, 240=2·380211	,, 300=2·477121
,, 4=0·602060	,, 5=0·698970
6·237543	,, 3=0·477121
	6·936513

Subtract sum of logs. of the first terms=6·237543
The answer. Log. of 5=0·698970

(3.) 24 boys : 27 men : : 10 hours.
45 days : 14 days
Ratio of work 1 : 2

Log. 24=1·380211	Log. 27=1·431364
,, 45=1·653213	,, 14=1·146128
,, 1=0·000000	,, 2=0·301030
3·033424	,, 10=1·000000
	3·878522

Subtract sum of logs. of the first terms=3.033424
The answer. Log. of 7=0·845098

(4.) 9 days : 3 days : : 120 men.
15 ho. : 12 ho.
30 yds. : 50 yds.
2 ft. : 4.5 ft.
4 ft. : 6 ft.

Log. 9=0·954243	Log. 3=0·477121
,, 15=1·176091	,, 12=1·079181
,, 30=1·477121	,, 50=1·698970
,, 2=0·301030	,, 4·5=0·653213
,, 4=0·602060	,, 6=0·778151
4·510545	,, 120=2·079181
	6·765817

Subtract sum of logs. of the first terms=4·510545
The answer. Log. 180=2·255272

Involution and Evolution.

(1.) Log. of 1·055 =0·023252
6
$\therefore (1{\cdot}055)^6=1{\cdot}37883$ =0·139512

(2.)
$$\begin{array}{lr} \text{Log. of } 1{\cdot}125 & =0{\cdot}051153 \\ & 10 \\ \hline \therefore (1{\cdot}125)^{10}=3{\cdot}24736 & = {\cdot}511530 \\ \hline \end{array}$$

(3.)
$$\begin{array}{lr} \text{Log. of } 5{\cdot}125 & =0{\cdot}709694 \\ & 7 \\ \hline \therefore (5{\cdot}125)^{7}=92866{\cdot}2 & =4{\cdot}967858 \\ \hline \end{array}$$

(4.)
$$\begin{array}{lr} \text{Log. of } 2625 & =3{\cdot}419129 \\ & \frac{1}{2} \\ \hline \therefore (2625)^{\frac{1}{2}}=51{\cdot}2347 & =1{\cdot}709564 \\ \hline \end{array}$$

(5.)
$$\begin{array}{lr} \text{Log. of } 17 & =1{\cdot}230449 \\ & \frac{1}{3} \\ \hline \therefore (17)^{\frac{1}{3}}=2{\cdot}57128 & = {\cdot}410149 \\ \hline \end{array}$$

(6.)
$$\begin{array}{lr} \text{Log. of } 5386{\cdot}25 & =3{\cdot}731286 \\ & \frac{1}{10} \\ \hline \therefore (5386{\cdot}25)^{\frac{1}{10}}=2{\cdot}36118 & = {\cdot}373129 \\ \hline \end{array}$$

(7.)
$$\begin{array}{lr} \text{Log. of } 2 & =0{\cdot}301030 \\ \text{Log. of } 7 & =0{\cdot}845098 \\ \hline \text{Log. of } \frac{2}{7} & =\bar{1}{\cdot}455932 \\ & \frac{1}{3} \\ \hline \therefore (\frac{2}{7})^{\frac{1}{3}}={\cdot}658633 & =\bar{1}{\cdot}818644 \\ \hline \end{array}$$

(8.)
$$\begin{array}{lr} \text{Log. of } \frac{1}{5}={\cdot}2 & =\bar{1}{\cdot}301030 \\ & \frac{1}{5} \\ \hline \therefore (\frac{1}{5})^{\frac{1}{5}}={\cdot}72478 & =\bar{1}{\cdot}860206 \\ \hline \end{array}$$

(9.)
$$\begin{array}{lr} 2 \text{ Log. of } 17=\text{Log. } 17^{2} & =2{\cdot}460898 \\ \text{Log. of } {\cdot}7854 & =\bar{1}{\cdot}895091 \\ \hline & 2{\cdot}355989 \\ \text{Log. of } 180 & 2{\cdot}255273 \\ \hline & 0{\cdot}100716 \\ & \frac{1}{4} \\ \hline \therefore \left(\frac{17^{2}\times{\cdot}7854}{180}\right)^{\frac{1}{4}}=1{\cdot}05967 & =0{\cdot}025179 \\ \hline \end{array}$$

(10.)
$$\begin{array}{lr} \text{Log. of } 8 & =0{\cdot}903090 \\ \text{Log. of } 240 & =2{\cdot}380211 \\ \text{Log. of } 66 & =1{\cdot}819544 \end{array}$$

ar. co. Log. of 19 =8·721246
" " Log. of 52 =8·283997
2·108088
$\frac{2}{3}$
3)4·216176
$\therefore \left(\frac{8\times240\times66}{19\times52}\right)^{\frac{2}{3}}$=25·4327 = 1·405392

(11.) 12 Log. 1·05= Log.$(1{\cdot}05)^{12}$=0·254268
Log. 705 =2·848189
Log. 2 =0·301030
ar. co. Log. 7 =9·154902
$\therefore (1{\cdot}105)^{12}\times705\times\frac{2}{7}$=361·733=2·558389

(12.) Log. 13 =1·113943
Log. 14 =1·146128
Log. 15 =1·176091
Log. 16 =1·204120
ar. co. Log. 17 =8·769551
ar. co. Log. 18 =8·744727
ar. co. Log. 19 =8·721246
ar. co. Log. 20 =8·698970
$\bar{1}$·574776
$\frac{1}{6}$
$\therefore \left(\frac{13\times14\times15\times16}{17\times18\times19\times20}\right)^{\frac{1}{6}}$=·849433=$\bar{1}$·929129

KEY TO TRIGONOMETRY.

Page 24.

(1.) AC : BC=R : tan. A, (Art. 34), and
R : sec. A=AC : AB, (Art. 33.)
Ar. co. Log. AC, 300 = 7·522879
Log. BC 221 = 2·344392
Log. R =10·000000
Log. tan. 36° 22′ 40″ = 9·867271

Ar. co. Log. R =10·000000
Log. sec. 36° 22′ 40″ =10·094137
Log. AC, 300 = 2·477121
∴ *AB=372·613* = 2·571258

Also ∠B=90°—∠A=90°—36° 22′ 40″=53° 37′ 20″.

(2.)
$$\left.\begin{array}{r} R : \tan. 49^\circ\ 12' = \text{base} : \text{perp.} \\ \text{and } R : \sec. 49^\circ\ 12' = \text{base} : \text{hyp.} \end{array}\right\} \text{(Art. 33.)}$$

Ar. co. Log. R =10·000000
Log. tan. 49° 12′ =10·063900
Log. 560 = 2·748188
∴ perp. 648·766 = 2·812088
Ar. co. Log. R =10·000000
Log. sec. 49° 12′ =10·184807
Log. 560 = 2·748188
∴ hyp. 857·027 = 2·932995

(3.)
$$\left.\begin{array}{r} R : \cos. 32^\circ\ 14' = \text{hyp.} : \text{base} \\ \text{and } R : \sin. 32^\circ\ 14' = \text{hyp.} : \text{perp.} \end{array}\right\} \text{(Art. 33.)}$$

Ar. co. Log. R =10·000000
Log. cos. 32° 14′ = 9·927310
Log. 641 = 2·806858
∴ base 542·21 = 2·734168
Ar. co. Log. R =10·000000
Log. sin. 32° 14′ = 9·727027
Log. 641 = 2·806858
∴ perp. 341·889 = 2·533885

(4.)
$$\left.\begin{array}{r} R : \text{cosec. } 49^\circ = \text{perp.} : \text{hyp.} \\ \text{and } R : \ \cot. \ 49^\circ = \text{perp.} : \text{base} \end{array}\right\} \text{(Art. 33.)}$$

Ar. co. Log. R =10·000000
Log. cosec. 49° =10·122220
Log. 500 = 2·698970
∴ hyp. =662·506 = 2·821190
Ar. co. Log. R =10·000000
Log. cot. 49° = 9·939163
Log. 500 = 2·698970
∴ base =434·643 = 2·638133

(5.) Hyp. : perp. =R : sine of the ∠ at the base. (Art. 34).
and R : cos. ∠ at the base = hyp. : base. (Art. 33).

Ar. co. Log. 100 = 8·000000
Log. 60 = 1·778151
Log. R =10·000000
∴ ∠ at base =36° 52′ 11·4″ = 9·778151
and vertical angle =90°—36° 52′ 11·4″=53° 7′ 48·6″.
Ar. co. Log. R =10·000000
Log. cos. 36° 52′ 11·4″ = 9·903090
Log. 100 = 2·000000
∴ *base* =80 = 1·903090

(6.) Hyp. : base =R : cos. ∠ at the base. (Art. 34).
and R : sin. ∠ at the base = hyp. : perp. (Art. 33).

Ar. co. Log. 580 = 7·236572
Log. 361 = 2·557507
Log. R =10·000000
∴ ∠ at the base =51° 30′ 27″ = 9·794079
and vertical angle =90°—51° 30′ 27″=38° 29′ 33″.

Ar. co. Log. R =10·000000
Log. sin. 51° 30′ 27″ = 9·893589
Log. 580 = 2·763428
∴ perp 543·959 = 2·657017

Art. 39.

(1.) 248 : 215= sin. 74° : sin. ∠ opposite 215.
Sin. 74° : sin. 49° 33′ 20″=248 : the third side.

Ar. co. Log. 248 = 7·605548
Log. 215 = 2·332438
Log. sin. 74° = 9·982842
∴ ∠ opp. 215=56° 26′ 40″ = 9·920828
180°—(74°+56° 26′ 40″)=49° 33′ 20″, the other angle.

Ar. co. Log. sin. 74°= Log. cosec. 74° =10·017158
Log. sin. 49° 33′ 20″ = 9·881405
Log. 248 = 2·394452
∴ the third side =196·343 = 2·293015

(2.) 215 : 169= sin. 72° : sin. ∠ opposite 169.
Sin. 72° : sin. 59° 37′ 9″=215 : the third side.

Ar. co. Log. 215 = 7·667562
Log. 169 = 2·227887
Log. sin. 72° = 9·978206
∴ ∠ opp. 169=48° 22′ 51″ = 9·873655
180°—(72°+48° 22′ 51″)=59° 37′ 9″, the other angle.

Log. cosec. 72° =10·021794
Log. sin. 59° 37′ 9″ = 9·935852
Log. 215 = 2·332438
∴ the third side =195·022 = 2·290084

(3.) 180°—(64° 13′+45° 27′)=70° 20′, the third angle.
Sin. 45° 27′ : sin. 64° 13′=1046 : side op. 64° 13′.
Sin. 45° 27′ : sin. 70° 20′=1046 : side op. 70° 20′.

Log. cosec. 45° 27′ =10·147131
Log. sin. 64° 13′ = 9·954457
Log. 1046 = 3·019532
∴ *the side op. 64° 13′*=1321·66 = 3·121120

Log. cosec. 45° 47′ =10·147131
Log. sin. 70° 20′ = 9·973897
Log. 1046 = 3·019532
∴ the side op. 70° 20′=1382·17 = 3·140560

(4.) Third angle =180—(49° 15′+70° 18′)=60° 27′.
Sin. 60° 27′ : sin. 70° 18′=5230 : side op. ∠70° 18′,
and sin. 60° 27′ : sin. 49° 15′=5230 : side op. ∠49° 15′.
Log. cosec. 60° 27′ =10·060518
Log. sin. 70° 18′ = 9·973807
Log. 5230 = 3·718502
∴ the side op. ∠70° 18′=5660·14 = 3·752827

Log. cosec. 60° 27′ =10·060518
Log. sin. 49° 15′ = 9·879420
Log. 5230 = 3·718502
∴ the side op. ∠49° 15′=4554·5 = 3·658440

(5.) 605 : 800 sin. 37° 14′ : sin. ∠ opposite 800.
Ar. co. Log. 605 =7·218245
Log. 800 =2·903090
Log. sin. 37° 14′ =9·781800
∴ the angle op. 800 { 53° 8′ 17″
is either (Art. 38), { or 126° 51′ 43″ =9·903135

Hence the third angle is either 180°—(37° 14′+53° 8′ 17″)=89° 37′ 43″; or 180°—(37° 14′+126° 51′ 43″)=15° 54′ 17″; and therefore

Sin. 37° 14′ : sin. 89° 37′ 43″=605 : the third side,
or Sin. 37° 14′ : sin. 15° 54′ 17″=605 : the third side.
Ar. co. Log. sin.=Log. cosec. 37° 14′=10·218200
Log. sin. 89° 37′ 43″ = 9·999991
Log. 605 = 2·781755
∴ the third side =999·88 = 2·999946
Log. cosec. 37° 14′ =10·218200
Log. sin. 15° 54′ 17″ = 9·437812
Log. 605 = 2·781755
Or the third side =274·01 = 2·437767

(6.) The third will be 180°—(90° 33′ 26″+39° 39′ 20″)=49° 47′ 14″:—and therefore
Sin. 90° 33′ 26″ : sin. 39° 39′ 20″=300·2 : one side;
And sin. 90° 33′ 26″ : sin. 49° 47′ 14″=300·2 : the other side.
Log. cosec. 90° 33′ 26″ =10·000021
Log. sin. 39° 39′ 20″ = 9.804937
Log. 300·2 = 2·477411
Side op. ∠39° 39′ 20″=191·588= 2·282369

Log. cosec. $90° 33' 26''$ $=10·000021$
Log. sin. $49° 47' 14''$ $=9·882895$
Log. $300·2$ $=2·477411$
Side op. $\angle 49° 47' 14''=229·26$ $=2·360327$

Art. 40, page 29th.

(1.) $241+173=414$ the sum of the sides.
$241-173=68$ the dif. of the sides.
$\frac{1}{2}(180°-103°)=38° 30'=\frac{1}{2}\angle(B+A)$;
$\therefore 414 : 68=\text{tan. } 38° 30' : \text{tan. } \frac{1}{2}(B-A)$.
Ar. co. Log. 414 $=7·383000$
Log. 68 $=1·832509$
Log. tan. $38° 30'$ $=9·900605$
Log. tan. $\frac{1}{2}(B-A)=7° 26' 37''$ $=9·116114$

Therefore $\angle B=45° 56' 37''$;
and $\angle A=31° 3' 23''$.
Hence sin. $31° 3' 23''$: sin. $103°=173 : AB$.
Log. cosec. $31° 3' 23''$ $=10·287450$
Log. sin. $103°$ $=9·988724$
Log. 173 $=2·238046$
$\therefore AB=326·753$ $=2·514220$

(2.) $79+67=146$ the sum of the sides.
$79-67=12$ the dif. of the sides.
$\frac{1}{2}(180°-85° 16')=47° 22'=\frac{1}{2}\angle(B+A)$.
$\therefore 146 : 12=\text{tan. } 47° 22' : \text{tan. } \frac{1}{2}(B-A)$.
Ar. co. Log. 146 $=7·835647$
Log. 12 $=1·079181$
Log. tan. $47° 22'$ $=10·035919$
Log. tan. $\frac{1}{2}(B-A)=5° 6' 6''$ $=8·950747$
$\therefore \angle B=52° 28' 6''$
And $\angle A=42° 15' 54''$:—and hence,
Sin. $42° 15' 54''$: sin. $85° 16'=BC : AB$
Log. cosec. $42° 15' 54''$ $=10·172269$
Log. sin. $85° 16'$ $=9·998516$
Log. $BC=67$ $=1·826075$
$\therefore AB=99·2795$ $=1·996860$

(3.) $4025+3450=7475$ the sum of the sides.
$4025-3450=575$ the dif. of the sides.
$\frac{1}{2}(180°-91° 30')=44° 15'=\frac{1}{2}\angle(A+B)$;
$\therefore 7475 : 575=\text{tan. } 44° 15' : \text{tan.} \frac{1}{2}(A-B)$.

Ar. co. Log. 7475 =6·126389
Log. 575 =2.759668
Log. tan. 44° 15′ =9·988629
Log. tan.$\frac{1}{2}$(A—B)=4° 17′ 7·7″ =8·874686
∴ ∠A=48° 32′ 7·7″;
And ∠B=39° 57′ 52·3″:—and hence,
Sin. 48° 32′ 7·7″ : sin. 91° 30′=BC : AB.
Log. cosec. 48° 32′ 7·7″ =10·125306
Log. sin. 91° 30′ = 9·999851
Log. BC=4025 = 3·604766
∴ AB=5369·37 = 3·729923

(4.) 800+749=1549 the sum of the sides.
800—749= 51 the dif. of the sides.
$\frac{1}{2}$(180°—80° 10′)=49° 55′=$\frac{1}{2}$∠(C+B);
∴ 1549 : 51=tan. 49° 55′ : tan.$\frac{1}{2}$(C—B).
Ar. co. Log. 1549 = 6·809949
Log. 51 = 1·707570
Log. tan. 49° 55′ =10·074904
Log. tan.$\frac{1}{2}$(C—B)=2° 14′ 25·4″= 8.592423
∴ ∠C=52° 9′ 25·4″
And ∠B=47° 40′ 34·6″:—and hence,
Sin. 52° 9′ 25·4″ : sin. 80° 10′=AB : BC.
Log. cosec. 52° 9′ 25·4″ =10·102540
Log. sin. 80° 10′ = 9·993572
Log. AB=800 = 2·903090
∴ BC=998·164 = 2·999202

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(1.)
AB, c=100
BC, a=150
AC, b=120
2|370
S=185 ar.co.Log.+10=17·732828
S—a= 35 Log. = 1·544068
S—b= 65 Log. = 1·812913
S—c= 85 Log. = 1·929419
2|23·019228
Constant=11·509614
Tan.$\frac{1}{2}$A=42° 43′ 47″ = 9·965546 {con.—Log.(S—a)}
Tan.$\frac{1}{2}$B=26° 26′ 44″ = 9·696701 {con.—Log.(S—b)}
Tan.$\frac{1}{2}$*C=20° 49′ 29″* = 9·580195 {con.—Log.(S—c)}
Proof sum 90° 0′ 0″ = half the sum of the angles.

Therefore ∠A=85° 27′ 34″, ∠B=52° 53′ 28″, and ∠C= 41° 38′ 58″.

(2.)
BC, $a=$ 398
AC, $b=$ 420
AB, $c=$ 470
2|1288
$S=$ 644 ar.co.Log. + 10=17·191114
$S-a=$ 246 Log. = 2·390935
$S-b=$ 224 Log. = 2·350248
$S-c=$ 174 Log. = 2·240549
2|24·172846
Constant =12·086423
Tan.$\frac{1}{2}$A=26° 22′ 54$\frac{1}{2}$″ = 9·695488 {con.–Log.$(S-a)$}
Tan.$\frac{1}{2}$B=28° 34′ 41″ = 9·736175 {con.–Log.$(S-b)$}
Tan.$\frac{1}{2}$C=35° 2′ 24$\frac{1}{2}$″ = 9·845874 {con.–Log.$(S-c)$}
Proof sum =90° 0′ 0″=half the sum of the angles.

Therefore ∠A=52° 45′ 49″, ∠B=57° 9′ 22″ and ∠C= 70° 4′ 49″.

(3.)
BC, $a=$140·4
AC, $b=$190·5
AB, $c=$260·1
2|591
$S=$295·5 ar.co.Log. + 10=17·529443
$S-a=$155·1 Log. = 2·190612
$S-b=$105· Log. = 2·021189
$S-c=$ 35·4 Log. = 1·549003
2|23·290247
Constant=11·6451235
Tan.$\frac{1}{2}$A=15° 53′ 45$\frac{1}{2}$″ = 9·4545115 {con.–Log. $(S-a)$}
Tan.$\frac{1}{2}$B=22° 48′ 53″ = 9·6239345 {con.–Log. $(S-b)$}
Tan.$\frac{1}{2}$C=51° 17′ 21$\frac{1}{2}$″ =10·0961205 {con.–Log. $(S-c)$}
Proof 90° 0′ 0″ =half sum of the angles

Therefore ∠A=31° 47′ 31″, ∠B=45° 37′ 46″, and ∠C= 102° 34″ 43″.

(4.)
BC, $a=$ 320
AC, $b=$ 800
AB, $c=$ 562
2|1682

$$\begin{array}{lcl}
S=841 \text{ ar.co.Log.}+10 & = & 17\cdot075204 \\
S-a=521 \text{ Log.} & = & 2\cdot716838 \\
S-b=\ \ 41 \text{ Log.} & = & 1\cdot612784 \\
S-c=279 \text{ Log.} & = & 2\cdot445604 \\
 & & 2\,|\,\overline{23\cdot850430} \\
\text{Constant} & = & \overline{11\cdot925215} \\
\text{Tan.}\tfrac{1}{2}\text{A}=\ \ 9^\circ\ 10'\ 42'' & = & 9\cdot208377\ \{\text{con.}-\text{Log.}\ (S-a)\} \\
\text{Tan.}\tfrac{1}{2}\text{B}=64^\circ\ \ 1'\ 54\tfrac{1}{2}'' & = & 10\cdot312431\ \{\text{con.}-\text{Log.}\ (S-b)\} \\
\text{Tan.}\tfrac{1}{2}\text{C}=16^\circ\ 47'\ 23\tfrac{1}{2}'' & = & 9\cdot479611\ \{\text{con.}-\text{Log.}\ (S-c)\} \\
\text{Proof}\quad \overline{90^\circ\ \ 0'\ \ 0''} & = & \text{half the sum of the angles.}
\end{array}$$

Therefore $\angle A=18^\circ\ 21'\ 24''$, $\angle B=128^\circ\ 3'\ 49''$, and $\angle C=33^\circ\ 34'\ 47''$.

Art. 50—53.

In these articles, after the expression for the sine or cosine of the sum or difference of the angles is reduced to the form of the sum or difference of two fractions, the next line is always obtained by dividing the numerator of each fraction by the hypotenuse of the triangle of which it is a side, and then multiplying this by the new divisor divided by the original denominator; this is evidently multiplying both numerator and denominator by the same quantity, by which the value of the fraction is not altered. The third line is then obtained by substituting the values of these fractions, as found by Arts. (6, 7.)

Art. 64.

$$\frac{\sin. A+\sin. B}{\cos. A+\cos. B}=\frac{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B).\quad (59.)}{2\cos.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B).\quad (61.)}$$

$$=\frac{\sin.\frac{1}{2}(A+B)}{\cos.\frac{1}{2}(A+B)}=\tan.\tfrac{1}{2}(A+B).\quad (13.)$$

Art. 65.

$$\frac{\sin. A+\sin. B}{\cos. B-\cos. A}=\frac{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B).\quad (59.)}{2\sin.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B).\quad (62.)}$$

$$=\frac{\cos.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A-B)}=\cot.\tfrac{1}{2}(A-B).\quad (14.)$$

Art. 66.

$$\frac{\sin. A-\sin. B}{\cos. B+\cos. A}=\frac{2\cos.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B).\quad (60.)}{2\cos.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B).\quad (61.)}$$

$$=\frac{\sin.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}(A-B)}=\tan.\tfrac{1}{2}(A-B).\quad (13.)$$

Art. 67.

$$\frac{\sin. A - \sin. B}{\cos. B - \cos. A} = \frac{2 \cos. \tfrac{1}{2}(A+B) \sin. \tfrac{1}{2}(A-B).\quad (60.)}{2 \sin. \tfrac{1}{2}(A+B) \sin. \tfrac{1}{2}(A-B).\quad (62.)}$$

$$= \frac{\cos. \tfrac{1}{2}(A+B)}{\sin. \tfrac{1}{2}(A+B)} = \cot. \tfrac{1}{2}(A+B). \quad (14.)$$

Art. 68.

$$\frac{\cos. B + \cos. A}{\cos. B - \cos. A} = \frac{2 \cos. \tfrac{1}{2}(A+B) \cos. \tfrac{1}{2}(A-B).\quad (61.)}{2 \sin. \tfrac{1}{2}(A+B) \sin. \tfrac{1}{2}(A-B).\quad (62.)}$$

$$= \frac{\cos. \tfrac{1}{2}(A+B)}{\sin. \tfrac{1}{2}(A+B)} \times \frac{\cos. \tfrac{1}{2}(A-B)}{\sin. \tfrac{1}{2}(A-B)}$$

$$= \cot. \tfrac{1}{2}(A+B) \cot. \tfrac{1}{2}(A-B).$$

which by (Art. 15) becomes either

$$\frac{\cot. \tfrac{1}{2}(A+B)}{\tan. \tfrac{1}{2}(A-B)}, \text{ or } \frac{\cot. \tfrac{1}{2}(A-B)}{\tan. \tfrac{1}{2}(A+B)}.$$

Art. 71.

The second and third values of $\cos. 2a$ are found from the first, by substituting $\cos.^2 a - 1$ for $-\sin.^2 a$ (21), and then $1 - \sin.^2 a$ for $\cos.^2 a$ (21).

Art. 74.

$$\frac{\sin. (a-b)}{\cos. (a-b)} = \frac{(\sin. a \cos. b - \cos. a \sin. b) \div \cos. a \cos. b}{(\cos. a \cos. b - \sin. a \sin. b) \div \cos. a \cos. b}$$

$$= \frac{\tan. a - \tan. b}{1 - \tan. a \tan. b}.$$

Art. 76.

$$\tan. 3a = \frac{\tan. a + \tan. 2a}{1 - \tan. a \tan. 2a} = \frac{\tan. a + \frac{2 \tan. a}{1 - \tan.^2 a}}{1 - \tan. a \times \frac{2 \tan. a}{1 - \tan.^2 a}}.$$

Multiply both numerator and denominator by $1 - \tan.^2 a$, and it becomes $\frac{\tan. a - \tan.^3 a + 2 \tan. a}{1 - \tan.^2 a - 2 \tan.^2 a} = \frac{3 \tan. a - \tan.^3 a}{1 - 3 \tan.^2 a}$.

Art. 77.

$$\tan. 4a = \frac{\tan. 2a + \tan. 2a}{1 - \tan.^2 2a} = \frac{\frac{2 \tan. a}{1 - \tan.^2 a} + \frac{2 \tan. a}{1 - \tan.^2 a}}{1 - \left(\frac{2 \tan. a}{1 - \tan.^2 a}\right)^2} =$$

$\frac{4 \tan. a - 4 \tan.^3 a}{1 - 6 \tan.^2 a + \tan.^4 a}$, by multiplying both numerator and denominator of the last fraction by $(1 - \tan.^2 a)^2$, or *$1 - 2 \tan.^2 a + \tan.^4 a$.*

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(1.) $$\text{Sin.}\,75^\circ=\text{sin.}\,(45^\circ+30^\circ)=\frac{1}{\sqrt{2}}\times\tfrac{1}{2}+\frac{1}{\sqrt{2}}\times\tfrac{1}{2}\sqrt{3}. \quad (51.)$$

$$=\frac{1}{2\sqrt{2}}(1+\sqrt{3})=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\text{Cos.}\,75^\circ=\text{cos.}(45^\circ+30^\circ)=\frac{1}{\sqrt{2}}\times\tfrac{1}{2}\sqrt{3}-\frac{1}{\sqrt{2}}\times\tfrac{1}{2}. \quad (52.)$$

$$=\frac{1}{2\sqrt{2}}(\sqrt{3}-1)=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$\text{Sin.}\,15^\circ=\text{sin.}\,(45^\circ-30^\circ)=\frac{1}{\sqrt{2}}\times\tfrac{1}{2}\sqrt{3}-\frac{1}{\sqrt{2}}\times\tfrac{1}{2}. \quad (53.)$$

$$=\frac{1}{2\sqrt{2}}(\sqrt{3}-1)=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$\text{Cos.}\,15^\circ=\text{cos.}\,(45^\circ-30^\circ)=\frac{1}{\sqrt{2}}\times\tfrac{1}{2}\sqrt{3}+\frac{1}{\sqrt{2}}\times\tfrac{1}{2}. \quad (54.)$$

$$=\frac{1}{2\sqrt{2}}\,(\sqrt{3}+1)=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

(2.) $$\text{Sin.}\,a=\frac{\text{sin.}\,a\ \text{cos.}\,a}{\text{cos.}\,a}=\text{cos.}\,a\times\frac{\text{sin.}\,a}{\text{cos.}\,a}=\text{cos.}\,a\ \text{tan.}\,a. \quad (13.)$$

$$=\frac{\text{cos.}\,a}{\frac{\text{cos.}\,a}{\text{sin.}\,a}}=\frac{\text{cos.}\,a}{\text{cot.}\,a}. \quad (14.)$$

$$=\frac{\text{sin.}\,a}{\text{cos.}\,a}\times\text{cos.}\,a=\text{tan.}\,a\times\frac{\text{cos.}\,a\div\text{cos.}\,a}{1\div\text{cos.}\,a}=\frac{\text{tan.}\,a}{\text{sec.}\,a}. \quad (20.)$$

$$=\frac{\text{sin.}\,a\times\text{cos.}\,a\times\frac{1}{\text{cos.}\,a}\div\text{sin.}\,a}{1\div\text{sin.}\,a}=\frac{\text{cos.}\,a\ \text{sec.}\,a}{\text{cosec.}\,a}.$$

Or, since (18) sin. $a=\frac{1}{\text{cosec.}\,a}$, and (17) cos. $a=\frac{1}{\text{sec.}\,a}$; whence cos. a sec. $a=1$. Substitute this value of 1 instead of it in the fraction $\frac{1}{\text{cosec.}\,a}$, and we have sin. $a=\frac{\text{cos.}\,a\ \text{sec.}\,a}{\text{cosec.}\,a}$.

$$\text{cos.}\,a=\frac{\text{sin.}\,a\ \text{cos.}\,a}{\text{sin.}\,a}=\text{sin.}\,a\,\frac{\text{cos.}\,a}{\text{sin.}\,a}=\text{sin.}\,a\ \text{cot.}\,a.$$

$$=\frac{\text{sin.}\,a\ \text{cos.}\,a\div\text{cos.}\,a}{\text{sin.}\,a\quad\div\text{cos.}\,a}=\frac{\text{sin.}\,a}{\text{tan.}\,a}$$

$$=\frac{\text{cos.}\,a\div\text{sin.}\,a}{1\quad\div\text{sin.}\,a}=\frac{\text{cot.}\,a}{\text{cosec.}\,a}$$

$$=\frac{\text{cos.}\,a\ \text{sin.}\,a\times\frac{1}{\text{sin.}\,a}\div\text{cos.}\,a}{1\div\text{cos.}\,a}=\frac{\text{sin.}\,a\ \text{cosec.}\,a}{\text{sec.}\,a}.$$

Note. The second value can also be obtained from the first by substituting $\frac{1}{\tan.}$ for cot. (15); and the fourth may be deduced from (17) and (18), which give cos. $a=\frac{1}{\sec. a}$ and $\frac{1}{\sin. a}=\text{cosec. } a$; ∴ $1=\sin. a \text{ cosec. } a$, and hence

$$\cos. a=\frac{\sin. a \text{ cosec. } a}{\sec. a}$$

$$\tan. a(13)=\frac{\sin. a}{\cos. a}=\frac{\sin. a \div \sin. a \cos. a}{\cos. a \div \sin. a \cos. a}=\frac{\sec. a}{\text{cosec. } a}.$$

$$=\frac{1}{\cot. a}(16)=\frac{\sin. a \text{ cosec. } a}{\cot. a}, \text{ for } \sin. a \text{ cosec. } a=1.$$

$$=\frac{\cos. a \sec. a}{\cot. a}, \text{ for } \cos. a \sec. a=1.$$

$$=\frac{1}{\cot. a}=\frac{\cos. a}{\frac{\sin. a \cos. a \cot. a}{\sin. a}}=\frac{\cos. a}{\sin. a \cot.^2 a}.$$

(3.) Since $A+B+C=180°$, $\sin. (A+B+C)=0$, (Arts. 30 and 70); but $\sin. (A+B+C)=\sin. (A+B) \cos. C+\cos. (A+B) \sin. C=(\sin. A \cos. B+\cos. A \sin. B) \cos. C+(\cos. A \cos. B-\sin. A \sin. B) \sin. C=\sin. A \cos. B \cos. C+\cos. A \sin. B \cos. C+\cos. A \cos. B \sin. C-\sin. A \sin. B \sin. C=0$, and therefore the sum of the plus terms is = the minus term, and hence by transposing we have

$\sin. A \cos. B \cos. C+\cos. A \sin. B \cos. C+\cos. A \cos. B \sin. C=\sin. A \sin. B \sin. C.$

Dividing both sides of the above equation by cos. A cos. B cos. C, it becomes

$$\tan. A+\tan. B+\tan. C=\tan. A. \tan. B. \tan. C.$$

Otherwise,

$$\tan. (A+B+C)=\frac{\tan. (A+B)+\tan. C}{1-\tan. (A+B) \tan. C}. \quad (73.)$$

$$=\frac{\frac{\tan. A+\tan. B}{1-\tan. A \tan. B}+\tan. C}{1-\frac{\tan. A+\tan. B}{1-\tan. A \tan. B}\tan. C}. \quad (73.)$$

Multiply both numerator and denominator by 1— tan. A tan. B, and we obtain

$$\tan. (A+B+C)=\frac{\tan. A+\tan. B+\tan. C-\tan. A \tan. B \tan. C}{1-\tan. A \tan. B-\tan. A \tan. C-\tan. B \tan. C}.$$

But $\tan. (A+B+C)=\frac{\sin. (A+B+C)}{\cos. (A+B+C)}=\frac{0}{-1}=0$, hence the numerator of the expression for the tangent must be $=0$; ∴ $\tan. A+\tan. B+\tan. C-\tan. A \tan. B \tan. C=0$; wherefore $\tan. A+\tan. B+\tan. C=\tan. A \tan. B \tan. C.$

Q. E. D.

(4.) $\text{Sin.}\ (60^\circ + \text{A}) - \text{sin.}\ (60^\circ - \text{A}) = 2\ \text{cos.}\ 60^\circ\ \text{sin.}\ \text{A}$ (60) $= \text{sin. A}$.

For by (Art. 84) $\text{cos.}\ 60^\circ = \frac{1}{2}$, hence $2\ \text{cos.}\ 60^\circ = 1$.

Again, $\text{cos.}(30^\circ - \text{A}) - \text{cos.}(30 + \text{A}) = 2\text{sin.}\ 30^\circ\ \text{sin. A}$. (62) $= \text{sin. A}$.

For by (Art. 84) $\text{sin.}\ 30^\circ = \frac{1}{2}$, hence $2\ \text{sin.}\ 30^\circ = 1$.

(5.) $\frac{1}{2}\left(\frac{\text{cos. A} + \text{sin. A}}{\text{cos. A} - \text{sin. A}}\right) + \frac{1}{2}\left(\frac{\text{cos. A} - \text{sin. A}}{\text{cos. A} + \text{sin. A}}\right) =$

reducing to a common denominator,

$$\frac{1}{2}\left(\frac{\text{cos.}^2\text{A} + 2\ \text{cos. A sin. A} + \text{sin.}^2\text{A}}{\text{cos.}^2\text{A} - \text{sin.}^2\text{A}}\right) +$$

$$\frac{1}{2}\left(\frac{\text{cos.}^2\text{A} - 2\ \text{cos. A sin. A} + \text{sin.}^2\text{A}}{\text{cos.}^2\text{A} - \text{sin}^2\text{A}}\right) =$$

performing the addition,

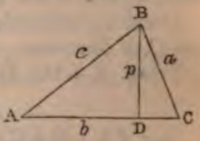
$$= \frac{\text{cos.}^2\text{A} + \text{sin.}^2\text{A}}{\text{cos.}^2\text{A} - \text{sin.}^2\text{A}} = \frac{1}{\text{cos. 2A}}, \text{ (Arts. 21 and 71.)}$$

$$= \text{sec. 2A}. \quad (20.)$$

(6.) Article 36 gives $\frac{a}{b} = \frac{\text{sin. A}}{\text{sin. B}}$, hence $\frac{a+b}{a-b} = \frac{\text{sin. A} + \text{sin. B}}{\text{sin. A} - \text{sin. B}}$,

(Alg. 108) and $\frac{\text{sin. A} + \text{sin. B}}{\text{sin. A} - \text{sin. B}} = \frac{\text{tan.}\ \frac{1}{2}(\text{A} + \text{B})}{\text{tan.}\ \frac{1}{2}(\text{A} - \text{B})}$. (59.)

hence $\frac{a+b}{a-b} = \frac{\text{tan.}\ \frac{1}{2}(\text{A} + \text{B})}{\text{tan.}\ \frac{1}{2}(\text{A} - \text{B})}$, and since a and b are the sides opposite to the angles A and B, the expression proves, that the sum of the sides is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference, which is the rule.



(7.) Let $(\text{AD} - \text{DC}) = d$ then,

$c - a : d = p\ \text{cosec. A} - p\ \text{cosec. C} : p\ \text{cot. A} - p\ \text{cot. C}\ (\div p)$.

$= \text{cosec. A} - \text{cosec. C} : \text{cot. A} - \text{cot. C}\ (\times\ \text{sin. A sin. C})$.

$= \text{sin. C} - \text{sin. A} : \text{sin. C cos. A} - \text{cos. C sin. A}$.

$= \text{sin. C} - \text{sin. A} : \text{sin. (C} - \text{A)}$ (53 .

$= 2\text{sin.}\frac{1}{2}(\text{C} - \text{A})\text{cos.}\frac{1}{2}(\text{C} + \text{A}) : 2\text{sin.}\frac{1}{2}(\text{C} - \text{A})\text{cos.}\frac{1}{2}(\text{C} - \text{A})$ (60 and 61.)

$= \text{cos.}\frac{1}{2}(\text{C} + \text{A}) : \text{cos.}\frac{1}{2}(\text{C} - \text{A})$. $(\div 2\ \text{sin}\frac{1}{2}.\text{C} - \text{A})$.

Again,

$c + a : d = p\ \text{cosec. A} + p\ \text{cosec. C} : p\ \text{cot. A} - p\ \text{cot. C}$.

$= \text{cosec. A} + \text{cosec. C} : \text{cot. A} - \text{cot. C}\ (\div p)$.

$\text{sin. C} + \text{sin. A} : \text{sin. C cos. A} - \text{cos. C sin. A}$.

$2\ \text{sin.}\ \frac{1}{2}\ (\text{C} + \text{A})\ \text{cos.}\ \frac{1}{2}\ (\text{C} - \text{A}) : 2\ \text{sin.}\ \frac{1}{2}\ (\text{C} - \text{A})\ \text{cos.}\ \frac{1}{2}\ (\text{C} - \text{A})$.

$= \text{sin.}\frac{1}{2}\ (\text{C} + \text{A}) \cdot \text{sin.}\frac{1}{2}\ (\text{C} - \text{A})$ $(\div 2\ \text{cos.}\frac{1}{2}\text{C} - \text{A})$.

Q. E. D.

Art. 87.

(1.) R : tan. 50° 30′=160 : the height above the eye.
Log. R =10·000000
Log. tan. 51° 30′ =10·099395
Log. 160 = 2·204120
Log. 201·148 = 2·303515
5 =height of the eye.
206·148 = „ tower.

(2.) R : tan. 47° 30′=100 : the height above the eye.
Log. R =10·000000
Log. tan. 47° 30′ =10·034948
Log. 100 = 2·000000
Log. 109·133 = 2·037948
5 =height of the eye.
114·133 = „ tower.

(3.) R : tan. 50° 43′=130 : height above the eye.
Log. R =10·000000
Log. tan. 50° 43′ =10·087244
Log. 130 = 2·113943
Log. 158·923 = 2·201187
5·5 =height of the eye.
164·423= „ tower.

Art. 88.

(1.) Sin. 131° 26′, or cos. 41° 26′ : sin. 38° 40′=266 : the one dist. and Sin. 131° 26′ : sin. 92° 46′=266 : the other distance.

Ar. co. Log. sin. 131° 26′=Log. cosec. 131° 26′ =10·125097
Log. sin. 38·40 = 9·795733
Log. 266 = 2·424882
Log. 221·672 = 2·345712
Ar. co. Log. sin. 131° 26′=Log. cosec. 131° 26′ =10·125097
Log. sin. 92° 46′=Log. cos. 2° 46′ = 9·999493
Log. 266 = 2·424882
Log. 354·382 = 2·549472

(2.) NE. ½ N. is 3½ points to the east of the north, and NW. ¼ W. is 4¼ points to the west of the north, the sum of these, or 7¾ points, is the angle formed by the ship's course, and the bearing of the headland. Again, the course the ship had when at the second station was SE. ¼ E., which is just 4 points from the bearing of the headland: hence the angles at the extremity of the base were 7¾ points and 4 points, *the sum* of which is 11¾ points, and since the *base is given 20 miles,* we have

Sin. $4\frac{1}{4}$ pts. : sin. 4 pts.=20 : Dist. from first station.
Sin. $4\frac{1}{4}$ pts. : sin. $7\frac{3}{4}$ pts.=20 : Dist. from second station.

Ar. co. Log. sin. $4\frac{1}{4}$ pts.=Log. cosec. $4\frac{1}{4}$ pts. =10·130210
Log. sin. 4 pts. = 9·849485
Log. 20 = 1·301030
Log. 19·0864, or 19·09 nearly = 12·80725
ar. co. Log. sin. $4\frac{1}{4}$ pts.=Log. cosec. $4\frac{1}{4}$ pts. =10·130210
Log. sin. $7\frac{3}{4}$ pts. = 9·999477
Log. 20 = 1·301030
Log. 26·96 = 1·430717

(3.) Sin. 60° : sin. 80°=260 : side op. 80°.
R : sin. 40°=side op. 80° : perp. breadth.

Compounding these proportions, we have
R sin. 60° : sin. 80° sin. 40°=260 : perp. breadth.
Hence, making R=1, and solving the proportion, we obtain *perp. breadth*= $\frac{260\times\text{sin. }80^\circ\text{ sin. }40^\circ}{\text{sin. }60^\circ}$.
=260 × sin. 80° sin. 40° cosec. 60°; hence

Log. cosec. 60° =10·062469
Log. sin. 40° = 9·808067
Log. sin. 80° = 9·993351
Log. 260 = 2·414973
Log. 190·046 = 2·278860

Art. 89.

			Distance.
Log. cosec. 10°	=10·760330	do.	10·760330
Log. sin. 59°	= 9·933066	cos.	9·711839
Log. sin. 49°	= 9·877780	do.	9·877780
Log. 50 feet	= 1·698970	do.	1·698970
186·27	= 2·270146	111·92	=2·048919

6 =height of the eye.
192·27 = „ tower.

(2.) 15° 32′ 18″—9° 56′ 26″=5° 35′ 52″
Log. cosec. 5° 35′ 52″ =11·010798
Log. sin. 15° 32′ 18″ = 9·427945
Log. sin. 9° 56′ 26″ = 9·237107
Log. 638 = 2·804821
Log. 302·463 = 2·480671

(3.) 123° 29′+50° 44′=174° 13′.

180°—174° 13′=5° 47′ the angle directly under the top of Arthur's Seat subtended by the distance of the stations, on the same horizontal plane as the base.

Height by the observed elevation at the western station.		Correction for the curvature of the earth (Note 4).	
Log. cosec. 5° 47′	=10·996682	do.	10·996682
Log. sin. 123° 29′	= 9·921190	do.	9·921190
Log. tan. 3° 59′	= 8·842825		
Log. 1410·42	= 3·149348	do.	3·149348
812·915	= 2·910045		4·067220
			2
5 height of the eye.			8·134440
		Constant	$\bar{8}\cdot378641$
3·259 correction for curvature.			=0·513081
821·174 height above the medium level of the tide.			

Height by the observed elevation at the eastern station.		Correction for the curvature of the earth (Note 4).	
Log. cosec. 5° 47′	=10·996682	do.	10·996682
Log. sin. 50° 44′	= 9·888858	do.	9·888858
Log. tan. 4° 17′ 30″	= 8·875315		
Log. 1410·42	= 3·149348	do.	3·149348
813·211	= 2·910203		4·034888
			2
5 height of the eye.			8·069776
		Constant	$\bar{8}\cdot378641$
2·808 correction for curvature.			= ·448417
821·019 height above the medium level of the tide.			

(4.)

Log. cosec. 71° 58′	=10·021876
Log. sin. 36° 22′	= 9·773018
Log. tan. 62° 36′	=10·285376
Log. 195·6	= 2·291369
Ans. 235·309	= 2·371639

Page 44.

(1.) In △BCD there are given ∠BCD, 37°; ∠BDC=∠ADB+∠ADC; ∴ ∠BDC=96°; hence the third angle CBD=47°, and the side CD=300 yards, to find CB and DB.

$$\left.\begin{array}{l}\text{Sin. CBD : sin. BDC} = \text{CD : CB}\\ \text{Sin. CBD : sin. BCD} = \text{CD : BD}\end{array}\right\}\text{By Art. 36.}$$

Log. cosec. 47°	=10·135873	Log. cosec. 47°	=10·135873
Log. sin. 96°	= 9·997614	Log. sin. 37°	= 9·779463
Log. CD, 300,	= 2·477121	Log. CD, 300	= 2·477121
Log. CB 407·95	= 2·610608	Log. BD 246·864	= 2·392457

Again, in the △ACD there are given ∠ADC, 41°;

∠ACD=∠ACB+∠BCD; ∴ ∠ACD=93°; hence the third angle CAD=46°, and the side CD=300 yards, to find AC and AD.

$$\left.\begin{array}{l}\text{Sin. CAD : sin. ACD=CD : AD}\\ \text{Sin. CAD : sin. ADC=CD : AC}\end{array}\right\}\text{By Art. 36.}$$

Log. cosec. 46°	=10·143066	Log. cosec. 46°	=10·143066
Log. sin. 93°	= 9·999404	Log. sin. 41°	= 9·816943
Log. CD 300	= 2·477121	Log. CD, 300,	= 2·477121
Log. AD 416·477=	2·619591	Log. AC, 273·609=	2·437130

Next, in the △ACB there are given the side AC=273·609, and CB=407·95, and the contained ∠ACB=56°, to find AB $\frac{1}{2}$(180°—56°)=26°, which is therefore half the sum of the angles at A and B. Hence by Art. 40,

CB=407·95
AC=273·609
Ar. co. Log. 681·559, sum of sides, =17·166496
Log. 134·341, dif. of sides, = 2·128208
Log. tan. 62° =10·274326
Log. tan. 20° 20′ 24″ = 9·569030
∠CAB =82° 20′ 24″
∠CBA =41° 39′ 36″

Lastly, to find the side AB from △ABC.

Log. cosec. CAB=82° 20′ 24″=10·003893
Log. sin. 56° = 9·918574
Log. CB=407·95 = 2·610608
Log. AB=341·252 = 2·533075

The same by the method of rectangular co-ordinates.

Vertex of the triangles.	Angles at C.	Angles at D.	Sum of angles.	Dif. of angles.	Perpendicular.	Distance from middle of base.
A	93°	41°	134°	52°	273·227	—164·319
B	37°	96°	133°	59°	245·511	175·804

For the perpendicular from A.			For its distance from the middle of the base.	
Log. 300	= 2·477121		Log. $\frac{1}{2}$(300)	= 2·176091
Log. cosec. 134°	=10·143066	=	Do.	=10·143066
Log. sin. 93°	= 9·999404		Log. sin. 52°	= 9·896532
Log. sin. 41°	= 9·816943		164·319	= 2·215689
273·227	= 2·436534			

From B.

Log. 300 = 2·477121 Log. $\frac{1}{2}$(300) = 2·176091
Log. cosec. 133° = 10·135873 = Do. = 10·135873
Log. sin. 37° = 9·779463 Log. sin. 59° = 9·933066
Log. sin. 96° = 9·997614 175·804 = 2·245030
245·511 = 2·390071

Dist. of the perpendiculars 340·123 Log. = 2·531636
: dif. of the perpendiculars 27·716 Log. + 10 = 11·442731
Log. tan. 4° 39′ 31″ = 8·911095
Log. sec. 4° 39′ 31″ = 10·001437
Log. 340·123 = 2·531636
∴ AB = 341·25 as before = 2·533073

(2.)

Vertex of the triangles.	Angles at C.	Angles at D.	Sum of angles.	Dif. of angles.	Perpendicular.	Distance from middle of base.
A	89° 18′	46° 14′	135° 32′	43° 4′	395·848	—187·163
B	48° 12′	87° 4′	135° 16′	38° 52′	406·202	171·186

Perpendicular from A. Distance of perp. from the middle of the base.

Log. 384 = 2·584331 Log. 192 = 2·283301
Log. cosec. 135° 32′ = 10·154595 = Do. = 10·154595
Log. sin. 89° 18′ = 9·999968 Log. sin. 43° 4′ = 9·834325
Log. sin. 46° 14′ = 9·858635 187·163 = 2·272221
395·848 = 2·597529

From B.

Log. 384 = 2·584331 Log. 192 = 2·283301
Log. cosec. 135° 16′ = 10·152546 = Do. = 10·152546
Log. sin. 48° 12′ = 9·872434 Log. sin. 38° 52′ = 9·797621
Log. sin. 87° 4′ = 9·999431 171·186 = 2·233468
406·202 = 2·608742

Since the perpendiculars fall on opposite sides of the middle of the base, the sum of the distances is the distance of the perpendiculars = 358·349, and the difference of the perpendiculars is 10·354; hence, by (Art. 95),

Log. 10·354 + 10 = 11·015108
Log. 358·349 = 2·554306
Log. tan. 1° 39′ 18″ = 8·460802

Log. sec. 1° 39′ 18″ =10·000181
Log. 358·349 = 2·554306
Ans. AB 358·5 = 2·554487

(3.)

Vertex of the triangles.	Angles at C.	Angles at D.	Sum of angles.	Dif. of angles.	Perpendicular.	Distance from middle of base.
A	92° 15′	44° 30′	136° 45′	47° 45′	367·979	—194·458
B	38° 45′	111° 50′	150° 35′	73° 5′	425·871	350·624

For the perpendicular from A. Distance of the perp. from the middle of the base.

Log. 360 = 2·556303 Log. 180 = 2·255273
L. cosec. 136° 45′=10·164193 =Do. =10·164193
Log. sin. 92° 15′ = 9·999665 Log. sin. 47° 45′= 9·869360
Log. sin. 44° 30′ = 9·845662 194·458 = 2·288826
367·979 = 2·565823

From B.

Log. 360 = 2·556303 Log. 180 = 2·255273
Log. cos. 150° 35′=10·308780 =Do. =10·308780
Log. sin. 38° 45′ = 9·796521 Log. sin. 73° 5′= 9·980789
Log. sin. 111 °50′= 9·967674 350·624 = 2·544842
425·871 = 2·629278

Since the perpendiculars fall on opposite sides of the middle of the base, the sum of the distances is the distance of the perpendiculars =545·082; and the difference of the perpendiculars is 57·892; hence (Art. 95)

Log. 57·892+10 =11·762619
Log. 545·082 = 2·736463 Do. = 2·736463
Log. tan. 6° 3′ 45″ = 9·026156 sec. =10·002436
Ans. AB 548·149 = 2·738899

(4.)

Castle Tower.

Log. 5675 = 3·753966 Log. $\frac{1}{2}$(5675) = 3·452936
L. cosec. 135° 57′=10·157837 =Do. =10·157837
Log. sin. 30° 22′ = 9·703749 Log. sin. 73° 13′= 9·981095
Log. sin. 105° 35′= 9·983735 3907·2 = 3·591868
3974·5 = 3.599287

Dome of St George's Church.

Log. 5675	= 3·753966	Log. ½(5675)	= 3·452936
L. cosec. 153°34½′	=10·351615	=Do.	=10·351615
L. sin. 25° 40′	= 9·636623	L. sin. 102° 14½′	= 9·990011
L. sin. 127° 54½′	= 9·897074	6231·1	= 3·794562
4357·9	= 3·639278		

Spire of St Andrew's Church.

Log. 5675	= 3·753966	Log. ½(5675)	= 3·452936
L. cosec. 149°41½′	=10·297007	=Do.	=10·297007
L. sin. 13° 57′	= 9·382152	L. sin. 121°47½′	= 9·929403
L. sin. 135° 44½′	= 9·843790	4779·1	= 3·679346
1892	= 3·276915		

Melville's Monument.

Log. 5675	= 3·753966	Log. ½(5675)	= 3·452936
L. cosec. 148° 48′	=10·285648	=Do.	=10·285648
L. sin. 11° 23′	= 9·295286	L. sin. 126° 2′	= 9·907774
L. sin. 137° 25′	= 9·830372	4429·5	= 3·646358
1463·1	= 3·165272		

North Leith Church Spire.

Log. 5675	= 3·753966	Log. ½(5675)	= 3·452936
L. cosec. 160° 4′	=10·467339	=Do.	=10·467339
L. sin. 28° 26′	= 9·677731	L. sin. 103°12′	= 9·988371
L. sin. 131° 38′	= 9·873560	8103	= 3·908646
—5923·7	= 3·772596		

Inchkeith Lighthouse.

Log. 5675	= 3·753966	Log. ½(5675)	= 3·452936
L. cosec. 170° 53½′	=10·800514	=Do.	=10·800514
L. sin. 57° 49′	= 9·927549	L. sin. 55° 15½′	= 9914729
L. sin. 113° 4½′	= 9·963784	14729	= 4·168179
—27913	= 4·445813		

Berwick-Law.

Log. 5675	= 3·753966	Log. ½(5675)	= 3·452936
Log. cosec. 177° 5′	=11·293423	=Do.	=11·293423
Log. sin. 115° 51	= 9·954213	Log. sin. 54° 37′	= 9·911315
Log. sin. 61° 14′	= 9·942795	—45465	= 4·657674
—87983	= 4·944397		

REMARK. The distance of any of the above places from Nelson's Monument may be obtained from the expression for the perpendicular, by adding together the first three logarithms in that expression, omitting the fourth; and its distance from Arthur's Seat may be obtained by adding together the first, second, and fourth logarithms, omitting the third, the sum in either case will be the logarithm of the distance, after omitting 20 from the index. The distance of any of the other objects from one another may be found by (Art. 95), as was shown in the last three examples.

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(1.) In the △DBC there are given ∠DBC=25° 45′, ∠DCB =30° 5′, and the side BC=290; to find BD.

$$\left.\begin{array}{ll}\text{Log. cosec. } 55^\circ\ 50' & =10{\cdot}082281\\ \text{Log. sin. } 30^\circ\ 5' & =\ 9{\cdot}700062\\ \text{Log. } 290 & =\ 2{\cdot}462398\end{array}\right\}\text{Art. 88.}$$
$$\text{BD}=175{\cdot}687 \qquad =\ 2{\cdot}244741$$

In the △ABC, find the angle ABC.

$$\left.\begin{array}{lll}\text{AC}=195 & & \\ \text{BC}=290 & \text{ar. co. Log.} & =\ 7{\cdot}537602\\ \text{AB}=240 & \text{ar. co. Log.} & =\ 7{\cdot}619789\\ \quad 2\,|\,\overline{725} & & \\ S=\overline{362.5} & \text{Log.}= & 2{\cdot}559308\\ S-b=167{\cdot}5 & \text{Log.} & 2{\cdot}224015\end{array}\right\}\text{Art. 45.}$$
$$2\,|\,\overline{19{\cdot}940714}$$

Log. cos.½ABC=20° 55′ 46″ = 9·970357
2
∠ABC =41° 51′ 32″
Now ∠ABD =∠ABC—∠DBC=16° 6′ 32″

Then in the △ABD there are given the sides AB=240, BD=175·687, and the ∠ABD=16° 6′32″ to find the other ∠s.

Ar. co. Log. (sum of the sides) 415·687 = 7·381233
Log. (dif. of the sides) 64·313 = 1·808299
Tan.½(sum of ∠s)= 81° 56′ 44″ =10·849213
Tan.½ (dif. of ∠s)= 47° 33′ 9″ =10·038745
Their difference ∠BAS= 34° 23′ 35″
∠ASB= 30° 5′
Wherefore ∠ABS=115° 31′ 25″

To find BS.		To find AS.	
Log. cosec. 30° 5′	=10·299938	Do.	=10·299938
Log. sin. 34°23′35″	= 9·751946	sin. 115°31′25″	= 9·955403
Log. 240	= 2·380211	Log. 240	= 2·380211
BS 270·46	= 2·432095	AS 432·07	= 2·635552

Lastly, in the △BSC, there are given ∠BSC=55° 50′ and

BC=290; also since ∠ABS=115° 31′ 25″ and ∠ABC=41° 51′ 32″, their difference, the ∠CBS=73° 39′ 53″.

Hence Log. cosec. 55° 50′ =10·082281
Log. sin. 73° 39′ 53″ = 9.982105
Log. BC 290 = 2·462398
CS 336·34 = 2·562784

(2.) In the △BDC there are given ∠DBC=33° 45′, ∠DCB=43°, and the side BC=790, therefore (Art. 88).

Log. cosec. 76° 45′ =10·011718
Log. sin. 33° 45′ = 9·744739
Log. 790 = 2·897627
BD=450·904 = 2·654084

Then in the △DBA′ since ∠DBA′=43°, half the sum of the ∠s BDA′ and BA′D=68° 30′, BD=450·904 and BA′=490; by Art. 40.

Ar. co. Log. (490+450·904)=940·904 = 7·026455
Log. (490—450·904)= 39·096 = 1·592133
Log. tan. 68° 30′ =10·404602
Log. tan. 6° 1′ 18″ = 9·023190
∴ ∠BDA′ =74° 31′ 18″,
and ∠BA′D =62° 28′ 42″=∠CA′S,
hence also ∠BA′S=117° 31′ 18″ Geo. prop. 1.
but the ∠A′SB =33° 45′;
∴ the ∠A′BS =28° 43′ 42″ Geo. prop. 19.

Therefore by Art. 39.

Log. cosec. 33° 45′ =10·255261 Do. =10.255261
Log. sin. 117° 31′ 18″= 9·947843 Log. sin. 28° 43′ 42′= 9.681836
Log. BA′=490 = 2·690196 Do. = 2·690196
BS=782·17 = 2·893300 A′S=423·93 = 2·627293

Lastly in the △A′SC by Art. 39.

Sin. A′SC : sin. SA′C=A′C : SC.
∴ Log. cosec. 43° =10·166217
Log. sin. 62° 28′ 42″ = 9·947843
Log, A′C=300 = 2·477121
SC=390·104 = 2·591181

(3.) First in the △DBC there are given by construction the ∠DBC=28° 24′, the ∠DCB=31°, consequently the ∠BDC=120° 36′, also the side BC=630, hence

sin. BDC : sin. DCB=BC : DB.
Log. cosec. 120° 36′ =10.065127
Log. sin. 31° = 9·711839
Log. BC=630 = 2·799341
DB =376·97 = 2·576307

In the △A″BC find the ∠A″BC (Art. 45).

A″C = 540
A″B = 500 ar. co. Log. = 7·301030
BC = 630 ar. co. Log. = 7·200659
2|1670
S = 835 Log. = 2·921686
S—*b* = 295 Log. = 2·469822
2|19·893197
Log. cos.½A″BC = 27° 50′ 6″ = 9·946598
2
∠A″BC = 55° 40′ 12″

Now ∠A″BD = ∠A″BC + ∠CBD = 84° 4′ 12″, and hence, in the △DBA″ there are given A″B = 500, DB = 376·97, and the ∠A″BD = 84° 4′ 12″, ∴ half the sum of the ∠s BDA″, and BA″D = 47° 57′ 54″; and therefore, by (Art. 40.)

ar.co.Log. (sum of the sides) 876·97 = 7·057015
Log. (dif. of the sides) 123·03 = 2·090009
Tan.½(sum of ∠s) = 47° 57′ 54″ = 10·045029
Tan.½(dif. of ∠s) = 8° 50′ 43″ = 9·192053
∴ ∠BDA″ = 56° 48′ 37″
and ∠BA″D = 39° 7′ 11″

Wherefore ∠BA″S = 140° 52′ 49″ and since ∠BSA″ = 31°,
∴ ∠A″BS = 8° 7′ 11″; and therefore in the
△A″BS : sin. S : sin. A″ = A″B : BS and
sin. S : sin. B = A″B : A″S

Log. cosec. 31° = 10·288161 Do. = 10·288161
Log. sin. 140° 52′ 49″ = 9·799991 Log. sin. 8° 7′ 11″ = 9·149964
Log. A″B = 500 = 2·698970 Do. = 2·698970
SB = 612·523 = 2·787122 SA″ = 137·118 = 2·137095

Lastly, in the △SBC, there are given ∠BSC = 59° 24′ ∠SBC = 63° 47′ 23″, and the side BC = 630.

∴ Log. cosec. 59° 24′ = 10·065127
Log. sin. 63° 47′ 23″ = 9·952879
Log. BC = 630 = 2·799341
SC = 656·67 = 2·817347

(4.) In the △BDC, there are given by construction the ∠DBC = 69° 30′, the ∠BCD = 54° 45′, ∴ the ∠BDC = 55° 45′, and the side BC = 620; hence sin. D : sin. C = BC : BD.

Log. cosec. 55° 45′ = 10·082710
Log. sin. 54° 45′ = 9·912031
Log. BC = 620 = 2·792392
BD = 612·538 = 2·787133

Next, in △ABC there are given the three sides to find *the* ∠*ABC* (Art. 45).

AC = 460
AB = 570 ar. co. Log. = 7·244125
BC = 620 ar. co. Log. = 7·207608
$2\underline{|1650}$
S = 825 Log. = 2·916454
$S-b$ = 365 Log. = 2·562293
$2\underline{|19·930480}$
Log. cos. $\frac{1}{2}$ABC = 22° 37′ 9″ = 9·965240
2
$\angle$ABC = 45° 14′ 18″

Now $\angle$ABD = $\angle$ABC + $\angle$CBD = 114° 44′ 18″, and hence in the $\triangle$ABD there are given AB = 570, BD = 612·538, and the $\angle$ABD = 114° 44′ 18″; ∴ half the sum of the $\angle s$ BAD and BDA = 32° 37′ 51″; wherefore by (Art. 40.)

ar. co. Log. (sum of the sides) 1182·538 = 6·927185
Log. (dif. of the sides) 42·538 = 1·628777
Log. tan. $\frac{1}{2}$(sum of $\angle s$) = 32° 37′ 51″ = 9·806374
Log. tan. $\frac{1}{2}$(dif. of $\angle s$) = 1° 19′ 10″ = 8·362336
∴ $\angle$BAD = 33° 57′ 1″
and $\angle$BDA = 31° 18′ 41″

Again, in the $\triangle$ASB the $\angle$BAS = 33° 57′ 1″, and the $\angle$ASB = 125° 15′, ∴ the $\angle$ABS = 20° 47′ 59″, and the side AB = 570; hence sin. S : sin. A = AB : BS
and sin. S : sin. B = AB : AS

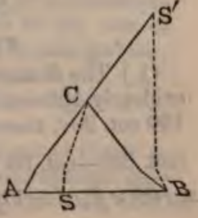
Log. cosec. 125° 15′	= 10·087969	Do.	= 10·087969
Log. sin. 33′ 57′ 1″	= 9·747002	Log. sin. 20° 47′ 59″	= 9·550353
Log. AB = 570	= 2·755875	Do.	= 2·755875
∴ BS = 389·804	= 2·590846	∴ AS = 247·855	= 2·394197

Lastly, in the $\triangle$BSC the $\angle$SBC = $\angle$ABC — $\angle$ABS = 24° 26′ 19″, and the $\angle$BSC = 124° 15′ therefore we have
sin. S : sin. B = BC : CS.

Log. cosec. 124° 15′ = 10·082710
Log. sin. 24° 26′ 19″ = 9·616704
Log. BC = 620 = 2·792392
∴ CS = 310·323 = 2·491806

(5.) From the $\triangle$ABC find all the angles by (Art. 49).

BC, a = 1000
AC, b = 974
AB, c = 1200
$2\underline{|3174}$



$S=1587$ ar.co.Log. $+10=16·799423$
$S-a=587$ Log. $=2·768638$
$S-b=613$ Log. $=2·787460$
$S-c=387$ Log. $=2·587711$
$2\,|\,24·943232$
Constant $=12·471616$

Log. tan. $\frac{1}{2}A=26°\ 46'\ 37\frac{1}{2}''=9·702978$ {cont—Log. $(s-a)$}
Log. tan. $\frac{1}{2}B=25°\ 47'\ 29''=9·684156$ {cont—Log. $(s-b)$}
Log. tan. $\frac{1}{2}C=37°\ 25'\ 53\frac{1}{2}''=9·889305$ {cont—Log. $(s-c)$}
$\therefore \angle A=53°\ 33'\ 15''$
$\angle B=51°\ 34'\ 58''$
$\angle C=74°\ 51'\ 47''$
Proof sum $=180°\ 0'\ 0''$ Geo. prop. 19.

Now, in the $\triangle CSB$ there are given the $\angle BSC=74°\ 12'$, the $\angle SBC=51°\ 34'\ 58''$, consequently the third $\angle SCB=54°\ 13'\ 2''$, and the side $BC=1000$; hence we have (Art. 39)

$$\sin. S : \sin. C = BC : BS.$$
$$\sin. S : \sin. B = BC : CS.$$

Log. cosec. 74° 12′	=10·016727	Do.	=10·016727
Log. sin. 54° 13′ 2″	= 9·909149	Log. sin. 51° 34′ 58″	= 9·894043
Log. BC=1000	= 3·000000	Do.	= 3·000000
∴ BS=843·094	= 2·925876	∴ CS=814·274	= 2·910770

Lastly, from $AB=1200$ subtract $BS=843·094$, and there remains $AS=356·906$.

(6.) The angles of the $\triangle ABC$ being the same as in the last example, the $\angle BCS'=180°-(\angle C\doteq 74°\ 51'\ 47'')=105°\ 8'\ 13''$, and since the $\angle CS'B$ is given $=37°\ 25'\ 53''$, $\therefore$ the third $\angle S'BC=37°\ 25'\ 54''$; therefore in the $\triangle S'BC$ there are given the three angles, and the side $CB=1000$, hence (Art. 39).

$$\sin. S' : \sin. B = CB : CS'$$
$$\sin. S' : \sin. C = CB : BS'$$

Log. cosec. 37° 25′ 53″	=10·216232	Do.	=10·216232
Log. sin. 37° 25′ 54″	= 9·783771	Log. sin. 105° 8′ 13″	= 9·984664
Log. CB=1000	= 3·000000	Do.	= 3·000000
∴ CS′ 1000	= 3·000003	∴ BS′=1588·16	= 3·200896

Hence AS' being $=AC+CS'=1974$.

PROMISCUOUS EXERCISES.

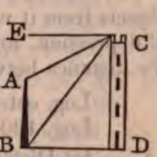
(1.) The distance of the first station from the tower is evidently 100 cot. 60°, and its distance from the second is 100 cot. 30°, therefore the distance of the stations is 100 (cot. 30°—cot. 60°) (art. 84) $=100\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)=100\left(\frac{3-1}{\sqrt{3}}\right)=100\times\frac{2}{3}\sqrt{3}=66\frac{2}{3}\sqrt{3}=115·47.$

Or, since the ∠AFE=60° (Fig. Art. 89) and the ∠AGF =30°, therefore the ∠GAF=30°, and hence the △AGF is isosceles, hence GF=AF; but if AE be made radius, AF will be the cosecant of the ∠AFE=60°, wherefore

R : cosec. 60°=AE : AF or GF.

∴ ar. co. Log. R =10·000000
Log. cosec. 60° =10·062469
Log. 100 = 2·000000
115·47 = 2·062469

(2.) Let CD represent the tower = 108, then the ∠ECB= the ∠CBD=60°, and since the ∠ECB=60°, and the ∠ECA =30, ∴ the ∠ACB=30°; also since the ∠ABD is a r^{t}∠, and the part CBD is= 60°, therefore the ∠ABC is =30°, hence the △ABC is isosceles, and the remaining ∠BAC=120°; now



△BCD gives R : sec. 30°=CD : CB, and
△ABC gives sin. 120° : sin. 30°=CB : AB, and compounding these proportions making R=1, and sin. 120°= sin. 60°, we have sin. 60° : sec. 30° sin. 30°=CD : AB, but

$$\text{sec. } 30^\circ \text{ sin. } 30^\circ = \frac{1}{\text{cos. } 30^\circ} \times \text{sin. } 30^\circ = \frac{\text{sin. } 30^\circ}{\text{cos. } 30^\circ} = \text{tan. } 30^\circ$$

hence sin. 60° : tan. 30°=CD : AB.

∴ Log. cosec. 60° =10·062469
Log. tan. 30° = 9·761439
Log. CD=108 = 2·033424
∴ AB=72 = 1·857332

Or, from the last proportion without tables, since

$$\text{sin. } 60^\circ = \tfrac{1}{2}\sqrt{3}, \text{ and tan. } 30^\circ = \frac{\sqrt{3}}{3} \text{ we have}$$

$$\tfrac{1}{2}\sqrt{3} : \tfrac{1}{3}\sqrt{3} = 108 : \text{AB}.$$

$$\therefore \text{AB} = \frac{\tfrac{1}{3}\sqrt{3} \times 108}{\tfrac{1}{2}\sqrt{3}} = \tfrac{2}{3} \times 108 = 72 \text{ Ans.}$$

(3.) Since the angles are to one another as the numbers 3, 4, and 5, the sum of these numbers is to any one of them, as 180° is to the angle corresponding to that number;

$$\left.\begin{array}{l} \text{hence } 12 : 3 = 180^\circ : 45^\circ \\ 12 : 4 = 180^\circ : 60^\circ \\ 12 : 5 = 180^\circ : 75^\circ \end{array}\right\} \text{the angles sought.}$$

Again, since the sines of the angles are proportional to the opposite sides (Art. 36); the sum of the natural sines *of these angles* will be to any one sine, as the sum of the sides is to the side opposite to the angle whose sine is used as the second term of the proportion.

sin. 45°= ·707107
sin. 60°= ·866025
sin. 75°= ·965926
Sum=2·539058 : 707107=2539·058 : 707·107
2·539058 : 866025=2539·058 : 866·025
2·539058 : 965926=2539·058 : 965·926

(4.) The tower being made radius, the distance of the objects from it will become the cotangents of their angles of depression, and the difference of the distances will be the distance between the objects. Hence,

Log. cot. 30°—10 = ·238561
Log. 100 = 2·000000
1st Distance, 173·205= 2·238561

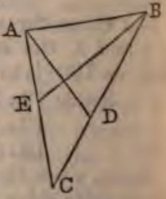
and Log. cot. 45°—10 = ·000000
Log. 100 = 2·000000
2d Distance, 100= 2·000000

∴ by subtraction 73·205= the distance of the objects.

NOTE. This exercise, as well as the next, can be more conveniently solved by using a table of natural cotangents, for the natural cotangent of the angle of depression, multiplied into the height of the tower, gives the distance of the object from the tower.

(5.) ar. co. Log. R =10·000000
Log. cot. 53° 24′ = 9·870793
Log. 120 = 2·079181
1st Distance, 89·12 = 1·949974

ar. co. Log. R =10·000000
Log. cot. 20° 16′ =10·432680
Log. 120 = 2·079181
2d Distance, 324·982 = 2·511861



(6.) Let A represent the left hand object, and B the right hand one, and C the first station; then the ∠C=38° 24′, ∠ADB=40° 13′, ∠AEB=39° 58′, also CD=300, and CE=350. Now, in the △ADC there are given the ∠C=38° 24′, the ∠D=139° 47′, being the supplement of the ∠ADB, and consequently the ∠CAD=1° 49′, also the side CD is given =300; hence sin. A : sin. D=CD : AC. (Art. 36.)

Log. cosec. 1° 49′ =11·498920
Log. sin. 139° 47′ = 9·810017
Log. 300 = 2·477121
AC=6110·24 = 3·786058

Also, in the △BEC there are given the ∠C=38° 24′, the ∠BEC=140°2′, ∴ the third ∠EBC=1° 34′, and the side EC=350; hence sin. B : sin. E=EC : CB.

Log. cosec. 1° 34′ =11·563200
Log. sin. 140° 2′ = 9·807766
Log. 350 = 2·544068
CB=8223·08 = 3·915034

Lastly, in the △ABC there are given AC, CB, and the ∠C to find the angles and the third side; hence (Art. 40) BC+AC : BC—AC= tan. $\frac{1}{2}$(A+B) : tan. $\frac{1}{2}$(A—B).

ar. co. Log. 14333·32 = 5·843653
Log. 2112·84 = 3·324867
Log. tan. 70° 48′ =10·458125
tan. $\frac{1}{2}$(A—B)=22° 56′ 34″ = 9·626645

Hence, the ∠A=93° 44′ 34″, and the ∠B=47° 51′ 26″, and sin. B : sin. C=AC : AB.

Log. cosec. 47° 51′ 26″ =10·129903
Log. sin. 38° 24′ = 9·793195
Log. 6110·24 = 3·786058
AB=5118·66 = 3·709156

NAVIGATION.

Art. 100.

(1.) Lat. left 42° 12′ N. Long. left 17° 18′ E.
Dif. lat. 150S=2° 30′ S. Dif.long.216W=3° 36′ W.
Lat. in 39° 42′ N. Long. in =13° 42′ E.

(2.) Lat. left 61° 24′ S. Long. left 49° 52′ W.
Dif. lat. 190 =3° 10′ N. Dif. long. 270= 4° 30′ W.
Lat. in 58° 14′ S. Long. in 54° 22′ W.

(3.) Lat. left 33° 18′ S. Long. left 18° 24′ E.
Dif. lat. 218= 3° 38′ S. Dif. long. 156 = 2° 36′ E.
Lat. in 36° 56′ S. Long. in 21° 0′ E.

(4.) Lat. left 12° 14′ N. Long. left 72° 16′ W.
Dif. lat. 512= 8° 32′ N. Dif. long. 724=12° 4′ E.
Lat. in 20° 46′ N. Long. in 60° 12′ W.

(5.)	Lat. left	29° 15′ N.	Long. left	25° 58′ E.
	Dif. lat. 147=	2° 27′ N.	Dif. long. 301	= 5° 1′ E.
	Lat. in	31° 42′ N.	Long. in	30° 59′ E.

(6.)	Lat. left	52° 18′ S.	Long. left	28° 27′ W.
	Dif. lat. 218=	3° 38′ S.	Dif. long. 186=	3° 6′ W.
	Lat. in	55° 56′ S.	Long. in	31° 33′ W.

Art. 101.

(1.)	Lat. from	25° 16′ N.	Long. from	15° 3′ W.
	Lat. at	20° 17′ N.	Long. at	42° 13′ W.
	Dif. lat.	4° 59′	Dif. long.	27° 10′
		60		60
		299 miles.		1630 miles.

(2.)	Lat. from	12° 50′ S.	Long. from	40° 34′ E.
	Lat. at	35° 42′ S.	Long. at	18° 14′ W.
	Dif. lat.	22° 52′	Dif. long.	58° 48′
		60		60
		1372 miles.		3528 miles.

(3.)	Lat. from	50° 17′ N.	Long. from	5° 7′ W.
	Lat. at	70° 12′ N.	Long. at	50° 16′ W.
	Dif. lat.	19° 55′	Dif. long.	45° 9′
		60		60
		1195 miles.		2709 miles.

(4.)	Lat. from	18° 25′ S.	Long. from	15° 35′ W.
	Lat. at	0° 0′	Long. at	35° 17′ W.
	Dif. lat.	18° 25′	Dif. long.	19° 42′
		60		60
		1105 miles.		1182 miles.

Art. 102.

(1.) Since S.W. by W. is 5 points, and the distance is 244 miles,

R : cos. 5 pts. =244 : Dif. lat.
R : sin. 5 pts. =244 : Departure.

∴ Log. R	=10·000000	Log. R	=10·000000
Log. cos. 5 pts.=	9·744739	Log. sin. 5 pts.=	9·919846
Log. 244	= 2·387390	Log. 244	= 2·387390
Dif. lat. 135·6S=	2·132129	Dept. 202·9	= 2·307236

Lat. sailed from	=49° 57′ N.
Dif. of Lat. 136 nearly=	2° 16′ S.
Lat. in	=47° 41′ N.

(2.) Since the latitude sailed from and the latitude arrived at are given, the difference of latitude made is easily found to be 271, and the course being 5 points, the following proportions will give the distance sailed and the departure made.

R : sec. 5 pts.=271 : Distance.
R : tan. 5 pts.=271 : Departure.

Log. R	=10·000000	Log. R	=10·000000
Log. sec. 5 pts.	=10·255261	Log. tan. 5 pts.	=10·175107
Log. 271	= 2·432969	Log. 271	= 2·432969
Dist. 487·8	= 2·688230	Dept. 405·6	= 2·608076

(3.) Here the course is $5\frac{3}{4}$ points, and the departure being given 406 miles; if the departure be made radius, the distance will be the cosecant of the course, and the difference of latitude will be the cotangent of the course; and hence, R : cosec. $5\frac{3}{4}$ pts.=406 : Distance.
R : cotan. $5\frac{3}{4}$ pts.=406 : Dif. Lat.

Log. R	=10·000000	Log. R	=10·000000
L. cosec. $5\frac{3}{4}$ pts.	=10·043837	L. cotan. $5\frac{3}{4}$ pts.	= 9·674829
L. 406	= 2·608526	L. 406	= 2·608526
Distance 449·1	= 2·652363	Dif. lat. 192 N.	= 2·283355

Lat. sailed from,	3° 15′ S.
Dif. lat. 192 miles	= 3° 12′ N.
∴ Latitude in	= 0° 3′ S.

(4.) Since the latitude sailed from was 2° 52′ S., and that arrived at 7° 23′ S., the difference of latitude made was 4° 31′=271 miles; we have by making the distance Radius,

488 : 271=R : cos. of the course,
and R : sin. course =488 : Departure.

Ar. co. Log. 488	= 7·311580	Log. R	=10·000000
Log. 271	= 2·432969	Log. sin. 56° 16′	= 9·919931
Log. R	=10·000000	Log. 488	= 2·688420
Log. cos. 56° 16′	= 9·744549	Dept. 405·8	= 2·608351

∴ the course was S. 56° 16′ E., and the departure 405·8 miles.

(5.) Making the distance Radius, the departure will be the sine of the course; and hence

137 : 112=R : sine of the course,
and R : *cos.* course =137 : dif. lat.

Ar. co. Log. 137 = 7·863279 Log. R =10·000000
Log. 112 = 2·049218 Log. cos. 54° 50′= 9·760390
Log. R =10·000000 Log. 137 = 2·136721
Log. sin. 54° 50′ nearly, = 9·912497 Dif. lat. 78·9 = 1·897111

Lat. sailed from, 50° 16′ N.
Dif. lat. 78·9 = 1° 19′ S.
Lat. in = 48° 57′.

(6.) 114 : 97=R : tan. course,
R : sec. course =114 : distance.

Ar. co. Log. 114= 7·943095 Log. R =10·000000
Log. 97 = 1·986772 Log. sec. =10·118308
Log. R =10·000000 Log. 114 = 2·056905
Tan. 40° 24′ = 9·929867 Distance 149·7 = 2·175213

Lat. sailed from, 32° 37′ N.
Dif. lat. 114 = 1° 54′ S.
Lat. in =30° 43′.

TRAVERSE TABLE I.

Corrected courses.	Points.	Dis-tances.	Dif. of Latitude.		Departure.	
			N.	S.	E.	W.
S.S.E. S.E. W. by S. W.N.W. S. by E.	2 4 7 6 1	15 34 16 39 40	 14·92 	13·85 24·04 3·12 39·23	5·74 24·04 7·80	 15·69 36·03
			14·92	80·24 14·92	37·58	51·72 37·58
			Dif. lat. =	65·32	Dept. =	14·14

Also Dif. lat. : Dept. =R : tan. course. (Art. 99.)

Ar. co. Log. 65·32 = 8·184954
Log. 14·14 = 1·150449
Log. R =10·000000
Tan. 12° 13′ = 9·335403

TRAVERSE TABLE II.

Corrected courses.	Points.	Distances.	Dif. of Latitude.		Departure.	
			N.	S.	E.	W.
S.¾W.	¾	40		39·57		5·87
S.S.E.¼E.	2¼	97		87·69	41·47	
N. by E.¼E.	1¼	72	69·84		17·49	
S.S.E.¾E.	2¾	108		92·63	55·52	
N. by E.¾E.	1¾	114	107·34		38·41	
S.E.¾S.	3¼	126		101·2	75·08	
N.N.E.¼E.	2¼	86	77·74		36·77	
			254·92	321·09	264·74	5·87
				254·92	5·87	
			Dif. lat. =	66·17	258·87	= Dept.

Since the latitude sailed from is 17° 58′ N., and the difference of latitude is 1° 6′ S.; therefore the latitude in is 16° 52′ N.

Also Dif. lat. : dept. = R : tang. course.

Ar. co. Log. 66·17 = 8·179339
Log. 258·87 = 2·413082
Log. R = 10·000000
Tan. 75° 40′ = 10·592421

TRAVERSE TABLE III.

Corrected courses.	Points.	Distances.	Dif. of Latitude.		Departure.	
			N.	S.	E.	W.
S.S.E.¼E.	2¼	16		14·46	6·84	
E.S.E.	6	23		8·80	21·25	
S.W. by W.½W.	5½	36		16·97		31·75
W.¾N.	7¼	12	1·76			11·87
S.E. by E.¼E.	5¼	41		21·08	35·17	
			1·76	61·31	63·26	43·62
				1·76	43·62	
			Dif. lat. =	59·55	19·64 =	Dept.

Now 59·55 miles of dif. lat. = 1°, which being south, subtract from 51° 25′ north, and the remainder is 50° 25′ *N., which is therefore* the latitude arrived at.

Also dif. lat. : dept. = R : tang. course,
and R : sec. course = dif. lat. : distance.

Ar. co. dif. lat. 59·55	= 8·225118	Ar. co. R	= 10·000000
Log. 19·64	= 1·293141	Log. sec. 18° 15′	= 10·022414
Log. R	= 10·000000	Log. 59·55	= 1·774882
Tang. 18° 15′	= 9·518259	Dist. 62·704	= 1·797296

Art. 105.

(1.) Log. sec. 42° 54′ — Log. R = ·135167
Log. 196 = 2·292256
Ans. 4° 27′ 36″ = 267·6 = 2·427423

(2.) Log. sec. 51° 25′ — Log. R = ·205058
Log. 1040 = 3·017033
Dif. long. 27° 47′ W. = 1667 = 3·222091
Sailed from, 9° 29′ W.
Ans. 37° 16′ W.

(3.) 735 : 456 = R : cos. lat.
Ar. co. Log. 735 = 7·133713
Log. 456 = 2·658965
Log. R = 10·000000
∴ the lat. = 51° 39′ 14″ = 9·792678

(4.) 240 : 308 = cos. 51° 16′ : cos. of the lat. sought.
Ar. co. Log. 240 = 7·619783
Log. 308 = 2·488551
Log. cos. 51° 16′ = 9·796364
∴ lat. sought = 36° 35′ 8″ = 9·904698

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(1.) Dif. lat. = 14° 46′ — 10° 30′ = 4° 16′ = 256 miles.
Middle lat. = $\frac{1}{2}$(14° 46′ + 10° 30′) = 12° 38′.
R : sec. course 3 pts. = dif. lat. : dist.
Log. sec. 3 pts. — R = ·080154
Log. dif. lat. 256 = 2·408240
∴ distance = 307·9 = 2·488394

Cos. mid. lat. : tan. course = dif. lat : dif. long. (7)

Ar. co. Log. cos. = Log. sec. 12° 38′ = 10·010644
Log. tan. 3 pts. = 9·824893
Log. 256 = 2·408240
∴ dif. long. = 175·7 = 2·243777

Hence the difference of longitude was 2° 56′ nearly, *which is east,* since the course is east, and since the longitude sailed from was west, their difference is the latitude in =21° 50′.

(2.) Dif. lat. =49° 54′—45° 8′=4° 46′=286 miles.
Middle lat. =½(49° 54′+45° 8′)=47° 31′.
320 : 286=R : cos. of the course.
Ar. co. Log. 320 =7·494850
Log. 286 =2·456366
Log. R =10·000000
∴ the course = 26° 39′ 6″ W. = 9·951216
cos. mid. lat. : sin. course = dist. : dif. long. (1.)
Ar. co. Log. cos. = Log. sec. 47° 31′ =10·170455
Log. sin. 26° 39′ 6″ = 9·651825
Log. 320 = 2·505150
∴ the dif. long. = 212·54 W. = 2·327430

This dif. long. being W., and =3° 33′ nearly, whilst that sailed from is also W., and =6° 19′, their sum, 9° 52′ W. is the longitude in.

(3.) 280 : 186=R : sin. of the course.
Ar. co. Log. 280 = 7·552842
Log. 186 = 2·269513
Log. R =10·000000
∴ The course=41° 37′ 39″ E. = 9·822355
R : cos. course=Dist. : dif. lat.
Ar. co. Log. R =10·000000
Log. cos. 41° 37′ 39″ = 9·873599
Log. 280 = 2·447158
∴ Dif. lat. 3° 29′ N.=209·3 = 2·320757
Lat. from 40° 3′ N.
∴ Lat. in 43° 32′ N.
Sum 83° 35′
Half sum 41° 47′ 30″ the middle latitude.
cos. mid. lat. : sin. course=Dist. : dif. long. (1.)
Ar. co. cos.=Log. sec. 41°47′ 30″ =10·127510
Log. sin. 41° 37′ 39″ = 9·822355
Log. 280 = 2·447158
∴ Dif. long. 4° 9′ E.=249·4 E. = 2·397023
Long. from 3° 52′ E.
Sum 8° 1′ E. the longitude in.

(4.) Dist. : dif. long.=cos. mid. lat. : sin. course. (4.)
Ar. co. Log. 500 = 7·301030
Log. 440 = 2·643453
Log. cos. 43° 45′ = 9·858756
∴ *The course*=39° 28′ 14″= 9·803239

R : cos. course = Dist. : dif. lat.

Ar. co. Log. R = 10·000000
Log. cos. 39° 28′ 14″ = 9·887590
Log. 500 = 2·698970
∴ Dif. lat. = 6° 26′ = 386 = 2·586560

Now, since the latitude sailed from was N., and the difference of latitude made was S., the latitude sailed from will be the middle latitude increased by *half* the difference of latitude made, and the latitude arrived at will be the middle latitude diminished by *half* the difference of latitude made; hence

The middle latitude	43° 45′ N.
Half the dif. lat.	3° 13′
Sum. lat. sailed from	46° 58′ N.
Dif. lat. arrived at	40° 32′ N.

KEY TO MENSURATION OF SURFACES.

Page 60.

(1.) 3525 links
Multiplied by 3525
17625
7050
17625
10575
124,25625 = 124 ac. 1 ro. 1 pole. Ans.
4
1,02500
40
1,00000

(2.) 5 ft. 7 in.
Multiplied by 3 5
16 9
2 3 11
19 0 11 Ans.

NOTE. It may be remarked here that in multiplying feet and inches by feet and inches, the denomination in the product next to *feet, though* commonly called inches, does not represent square *inches, but* a surface that is 12 inches long and 1 inch broad, being *the 12th* part of a square foot, whilst the second denomination from *feet, commonly* called parts, represents square inches.

(3.) $25 \times 17\frac{1}{2} = 437\frac{1}{2}$ square feet, which being divided by 9, the number of square feet in a square yard, gives $48\frac{11}{18}$ the Ans.

(4.)

$$\begin{array}{lr}
 & 1535 \text{ links} \\
\text{Multiplied by} & 1270 \\ \hline
 & 107450 \\
 & 3070 \\
 & 1535 \\ \hline
 & 19{,}49450 = 19 \text{ ac. } 1 \text{ ro. } 39{\cdot}12 \text{ poles.} \\
 & 4 \\ \hline
 & 1{,}97800 \\
 & 40 \\ \hline
 & 39{,}12000
\end{array}$$

(5.)

$$\begin{array}{lr}
 & 320 \text{ feet} \\
\text{Multiplied by} & 13\frac{1}{2} \\ \hline
 & 160 \\
 & 960 \\
 & 320 \\ \hline
36 \times 9 = 324) & 4320(13\frac{1}{3} \text{ roods. Ans.} \\
 & 324 \\ \hline
 & 1080 \\
 & 972 \\ \hline
 & \frac{108}{324} = \frac{1}{3}.
\end{array}$$

£2 12 6 $= 2\frac{5}{8} = £\frac{21}{8}$, and $13\frac{1}{3} = \frac{40}{3}$, hence

$£\frac{21}{8} \times \frac{40}{3} = £35$. Ans.

Or 1 rood : $13\frac{1}{3}$ roods : : £2 12 6 ; £35.

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$$\begin{array}{lr}
(1.)\ \text{Nat. sin. } 81°15' & = {\cdot}988362 \\
\text{Multiplied by } 736 \times 609 & = 448224 \\ \hline
 & 3953448 \\
 & 1976724 \\
 & 1976724 \\
 & 7906896 \\
 & 3953448 \\
 & 3953448 \quad \text{Ans.} \\ \hline
 & 4{,}43007{\cdot}569088 = 4 \text{ ac. } 1 \text{ ro. } 28{\cdot}812 \text{ p.} \\
 & 4 \\ \hline
 & 1{,}72030{\cdot}276352 \\
 & 40 \\ \hline
 & 28{,}81211{\cdot}05408
\end{array}$$

Or, by Logarithms.

Log. 56·75 =1·753966
Log. 22·5 =1·352183
Log. sin. 36° 45′ =9·776937
Ans.=763·986 =2·883086

(5.) nat. sin. 45° =·707107
$\frac{1}{2}(30\times24)$ = 360
42426420
2121321
9)254·558520
Sq. yards 28·28428 Ans.

Or by Logarithms.

Log. sin. 45° =9·849485
Log. 30 =1·477121
Log. 12 =1·079181
=2·405787
Subtract Log. 9= ·954243
Sq. yards. Ans. 28·2842 =1·451544

Page 63.

(1.) Half the sum of the sides $=\frac{234\times3}{2}=351=3\times117$, and each of the remainders $=351-234=117$; hence the area $=\sqrt{3\times117\times117\times117\times117}=117\times117\sqrt{3}=13689\times\sqrt{3}$. But the square root of 3=1·73205, which being multiplied by 13689, gives 23710 square links, which is equal to ·23710 acres.
4
·94840
40
38·93600 poles, the Ans.

Or by Logarithms.

Log. half sum of the sides =351 =2·545307
3 Log. 117 the remainder =6·204558
2)8·749865
Ans. Sq. links 23710=4·374932 Log. area.

(2.) $\frac{1}{2}(39+36+15)=$ 45 Log. =1·653213
45—39 = 6 „ = ·778151
45—36 = 9 „ = ·954243
45—15 =0 „ =1·477121
2)4·862728
Ans. 30 yds.=270 feet =2·431364 Log. area.

Or the sides expressed in yards are 13, 12, and 5, hence half their sum is 15, and the several remainders are 2, 3, and 10; therefore the area in square yards is $\sqrt{15\times2\times3\times10}$ $=\sqrt{30\times30}=30$ Ans.

(3.) Here the sides expressed in chains are 5, 5, and 6, therefore their half sum is 8, and the remainders are 3, 3, and 2; hence the area in square chains is $\sqrt{8\times3\times3\times2}$ $=\sqrt{144}=12$; and since 10 square chains make an acre, the area is 1·2 acres =1 acre 32 poles.

Or by Logarithms.

	500	
	500	
	600	
2	1600	
½ sum=	800 Log.	= 2·903090
	800—500=300 Log.	= 2·477121
	800—500=300 Log.	= 2·477121
	800—600=200 Log.	= 2·301030
		2|10·158362
	Sq. links 120000	= 5·079181
	Or 1·2 acres as before.	

(4.) Here ½(74+82+90)=123, and the three remainders are 49, 41, and 33; hence the area will be = $\sqrt{123\times49\times41\times33}$ = $\sqrt{815431}$, the square root of which is equal to 2855·61 the answer.

Or by Logarithms.

	Sides.	Remainders.	Logarithms.
	74	49	=1·690196
	82	41	=1·612784
	90	33	=1·518514
2	246		
½ sum	123		=2·089905
			2|6·911399
		Ans. 2855·61 feet	=3·455699

Page 6. Rule IV.

(1.) Here the sum of the angles 68° 24′ and 71° 13′ is *139° 37′, the cosecant of* which is the secant of its excess *above 90°, that is of 49° 37′.*

(2.) $\frac{557+493}{2}=\frac{1050}{2}=$ 525

$$\begin{array}{r}
525 \\
1476 \\
\hline
3150 \\
3675 \\
2100 \\
525 \\
\hline
7{,}74900 \text{ sq. links.} \\
4 \\
\hline
2{,}99600 \\
40 \\
\hline
39{\cdot}84000
\end{array}$$

Hence the Ans. = 7 acres, 2 roods, 39·84 poles.

(3.) In this exercise there are given in each of the triangles ABD and DBC two sides and the contained angle; hence we must use Rule II. Problem II., to find the area.

Log. 490 = 2·690196
Log. $(\frac{756}{2}=378)$ = 2·577492
Log. sin. 74° 44′ = 9·984397
Area of △ABD=178684 =5·252085

Log. 464 = 2·666518
Log. $(\frac{756}{2}=378)$ = 2·577492
Log. sin. 80° 17′ = 9·993725
Area of △DBC=172876 = 5·237735

∴ △ABD+△DBC=3,51560= 3 acres, 2 roods, 2·496 poles.

$$\begin{array}{r}
4 \\
\hline
2{,}06240 \\
40 \\
\hline
2{,}49600
\end{array}$$

4. Here in the triangles ABD and DBC there are given the three sides of each, the diagonal BD being common to both; hence their areas must be found by Rule III., Problem II.

To find the area of the △ABD.

AB=36
AD=44
BD=62
2|142

S or half sum of sides,	71	Log. of, =	1·851258
S—AB	=35	„ =	1·544068
S—AD	=27	„ =	1·431364
S—BD	= 9	„ =	0·954243
			2|5·780933
∴ area of ABD	=777·084	=	2·8904665

To find the area of the △DBC.

DB= 62
BC= 34
CD= 42
2|138

S or half sum of sides,	69	Log. of, =	1·838849
S—DB	= 7	„ =	0·845098
S—BC	= 35	„ =	1·544068
S—CD	= 27	„ =	1·431364
			2|5·659379
∴ area of △DBC	=675·6	=	2·8296895

Hence the sum of the areas is 777·084 + 675·6 = 1452·684, which being divided by 9, gives 161·409 sq. yards.

(5.) By Rule II. of this Problem we have

Log. 760	= 2·880814
Log. 810	= 2·908485
Log. sin. 79° 16′	= 9·992335
	15·781634
Subtract	10·301030
3 ac. 3·864 poles =3,02415	= 5·480604

4
·9660
40
3·86400

(6.) In the △ADC there are given the two sides AD=320 and DC=360, and the included ∠ADC=80° 30′; from which, by Trig., Art. 40, the diagonal AC can be found, and then there are given the three sides of each of the triangles ADC and ABC, from which the area can be found, *by Rule III.*, Problem II.; thus,

$360+320=680$, the sum of the sides,
$360-320=40$, the dif. of the sides,
$\frac{1}{2}(180^\circ-86^\circ\,30')=46^\circ\,45'=\frac{1}{2}(\angle DAC+\angle DCA)$;
$\therefore\ 680 : 40 =$ tan. $46^\circ\,45'$: tan.$\frac{1}{2}(\angle DAC-\angle DCA)$.
(Trig. 40.)

Ar. co. Log. 680 $= 7{\cdot}167491$
Log. 40 $= 1{\cdot}602060$
Log. tan. $46^\circ\,45'$ $=10{\cdot}026546$
Log. tan.$\frac{1}{2}(A-C)=3^\circ\,34'\,42'' = 8{\cdot}796097$

But by the last part of Note II., Art 40, of Trigonometry, cos. $3^\circ\,34'\,42''$: cos. $46^\circ\,45'=680 : AC$.

Ar. co. Log. cos. = Log. sec. $3^\circ\,34'\,42'' =10{\cdot}000847$
Log. cos. $46^\circ\,45'$ $= 9{\cdot}835807$
Log. 680 $= 2{\cdot}832509$
$\therefore\ AC=466{\cdot}834$ $= 2{\cdot}669163$

To find the area of the $\triangle ADC$.

$AD= 320$
$DC= 360$
$AC= 466{\cdot}834$
$2\,|\,1146{\cdot}834$

$S = 573{\cdot}417$ Log. of $= 2{\cdot}758471$
$S-AD= 253{\cdot}417$ „ $= 2{\cdot}403836$
$S-DC= 213{\cdot}417$ „ $= 2{\cdot}329229$
$S-AC= 106{\cdot}583$ „ $= 2{\cdot}027689$
$2\,|\,9{\cdot}519225$

$\therefore$ area of $\triangle ADC=57492{\cdot}7$ $= 4{\cdot}7596125$

To find the area of the $\triangle ABC$.

$AB= 534$
$BC= 510$
$AC= 466{\cdot}834$
$2\,|\,151{\cdot}834$

$S = 755{\cdot}417$ Log. of $= 2{\cdot}878187$
$S-AB= 221{\cdot}417$ „ $= 2{\cdot}345211$
$S-BC= 245{\cdot}417$ „ $= 2{\cdot}389905$
$S-AC= 288{\cdot}583$ „ $= 2{\cdot}460271$
$2\,|\,10{\cdot}073574$

$\therefore$ area of $\triangle ABC=108839$ $= 5{\cdot}036787$

Wherefore the area of the whole trapezium is $57492{\cdot}7+108839=1,66331{\cdot}7=1$ ac., 3 ro., 26·13 poles.

4
2,65326·8
40
26·13072

PROBLEM V.

(1.) Find the area of the trapezium ABCG, by multiplying AC by one-half the sum of the perpendiculars Bb, Gc. Again, to find the area of the trapezium FEDG, multiply FD by one-half the sum of Go and Ea; and find the area of the $\triangle$GDC from the three sides.

$$\begin{array}{lr}
\text{Area of ABCG} = 530 \times \dfrac{98+134}{2} & = 61480 \\
\text{Area of GFED} = 486 \times \dfrac{108+84}{2} & = 46656 \\
\text{Area of } \triangle\text{GDC} & = \underline{35789{\cdot}8} \\
\text{Ans. 1 acre, 1 rood, 30·2812 poles,} & = 1{,}43925{\cdot}8
\end{array}$$

Calculation of the area of the $\triangle$GDC.

$$\begin{array}{rrlr}
 & \text{GC} = & 394 & \\
 & \text{DC} = & 182 & \\
 & \text{GD} = & \underline{424} & \\
 & & |\overline{1000} & \\
S & = & 500 \text{ Log. of} & = 2{\cdot}698970 \\
S-\text{GC} & = & 106 \quad ,, & = 2{\cdot}025306 \\
S-\text{DC} & = & 318 \quad ,, & = 2{\cdot}502427 \\
S-\text{GD} & = & 76 \quad ,, & = \underline{1{\cdot}880814} \\
 & & & 2|9{\cdot}107517 \\
\multicolumn{3}{r}{\therefore \triangle\text{GDC} = 35789{\cdot}8} & = \overline{4{\cdot}553758{\cdot}5}
\end{array}$$

PROBLEM VI. RULE I.

(2.) $8 \times 5{\cdot}506 \times 2\frac{1}{2} = 110{\cdot}12$ sq. inches.

(3.) $15 \times 18{\cdot}11 \times 4 = 1086{\cdot}6$ sq. feet $= 120{\cdot}73$ sq. yds.

(4.) $4 \times 6{\cdot}156 \times 5 = 123{\cdot}12$ sq. feet.

Rules Second and Third, page 67.

$$\begin{array}{lr}
\text{(1.) Area of a pentagon whose side is } 1 = & 1{\cdot}720477 \\
\text{Multiply by } (15)^2 \hfill = & \underline{225} \\
 & 8602385 \\
 & 3440954 \\
 & \underline{3440954} \\
\text{Ans.} & 387{\cdot}107325
\end{array}$$

(2.) Tabular area of a hexagon = 2·598076
Multiply by $(20)^2$ = 400
9|1039·230400
Sq. yards, 115·47 Ans.

(3.) Tabular area of a heptagon = 3·633913
Multiplied by 3^2 = 9
Sq. feet, 32·705217 Ans.

(4.) Tabular area of an octagon = 4·828427
Multiplied by $(10)^2$ = 100
9|482·8427
Sq. yards, 53·6492 Ans.

(5.) Tabular area of a decagon = 7·694209
Multiplied by $(18)^2 \div 144$ = $2\frac{1}{4}$
1923552
15388418
Sq. feet, 17·31197 Ans.

(6.) Tabular area of a dodecagon = 11·196152
Multiplied by $(4\frac{1}{2})^2$ = $20\frac{1}{4}$
2·799038
223·923040
Sq. feet, 226·722078 Ans.

(7.) 2 Log. 2·5 = 0·795880
Log. 16 = 1·204120
Log. cot. $\left(\frac{180^\circ}{16} = 11^\circ\ 15'\right)$ =10·701338
Ans. =502·734 = 2·701338

(8.) Since 12 feet = 4 yards, half the side is 2 yards.
Hence 2 Log. 2 = 0·602060
Log. 100 = 2·000000
Log. cot. $\left(\frac{180^\circ}{100} = 1^\circ\ 48'\right)$ =11·502707
Ans. 12728·2 = 4·104767

NOTE. The first six examples can all be calculated by the same method as the seventh and eighth, and the results will be the same for the first six figures. The answers have only been given to that degree of accuracy which can be obtained from either of the rules.

PROBLEM VII. Page 68.

(1.) $3\cdot1416\times3 \qquad =9\cdot4248$ feet. Ans.

(2.) $3\cdot1416\times60 \qquad =188\cdot496$ feet. Ans.

(3.)

$$\begin{array}{r}
3\cdot1416 \\
7912 \\
\hline
62832 \\
31416 \\
282744 \\
219912 \\
\hline
\text{Ans. } 24856\cdot3392 \text{ miles.}
\end{array}$$

(4.)

$$\begin{array}{r}
3\cdot1416 \\
89170 \\
\hline
2199120 \\
31416 \\
282744 \\
251328 \\
\hline
\text{Ans. } 280136\cdot472 \text{ miles.}
\end{array}$$

(5.) Diameter when the circumference is $1=\cdot31831$
Multiply by 7 feet 3 inches $= 7\frac{1}{4}$ feet.

$$\begin{array}{r}
7958 \\
222817 \\
\hline
\text{Ans. } 2\cdot30775 \text{ feet.}
\end{array}$$

(6.) Diameter to circumference $1 = \cdot31831$
Multiply by 16

$$\begin{array}{r}
\hline
\text{Ans. } 5\cdot09296 \text{ feet.}
\end{array}$$

(7.) $\cdot31831\times300 \qquad =95\cdot493$ feet. Ans.

(8.) The number of feet in a mile is 5280, which being divided by the number of times the wheel turned, viz. 528, gives the circumference of the wheel 10 feet; hence $\cdot31831\times10=3\cdot1831$ feet, the diameter.

PROBLEM VIII.

(1.) $\dfrac{30\cdot6\times8-50\cdot8}{3}=64\cdot\dot{6}$. Ans.

(2.) $\dfrac{25\cdot5\times8-45}{3}=53$. Ans.

(3.) $\sqrt{64+16}=8{\cdot}94427$, chord of half the arc.

$$\frac{8{\cdot}94427\times8-16}{3}=18{\cdot}518.\quad \text{Ans.}$$

(4.) $\sqrt{144+81}=15$, the chord of half the arc.

$$\frac{15\times8-24}{3}=32.\quad \text{Ans.}$$

Page 70.

(1.) By Table XI. arc of 40° to radius 1 = ·6981317
„ 9° „ = ·1570796
„ 30′ „ = ·0087266
∴ arc of 49° 30′ to radius 1 = ·8539379
Multiply by 30
Ans. =25·618137

(2.) Since the chord of the arc is 20, and the radius of the circle is also 20, the sine of half the arc is $\frac{10}{20}=\frac{1}{2}=$ sin. 30°; therefore the whole arc is 60°; hence by Rule II. we have

·0174533
Multiplied by 60 × 20 = 1200
Ans. Length of the arc = 20·94396

(3.) By Note I., tan. $\frac{1}{4}$ arc. $=\frac{12}{18}=\frac{2}{3}$; and therefore Log. tan. $\frac{1}{4}$ arc. $=10{\cdot}301030-{\cdot}477121=9{\cdot}823909$; and hence $\frac{1}{4}$ the arc $=33°\ 41'\ 24\frac{1}{4}''$, which being multiplied by 4, gives the arc $=134°\ 45'\ 37''$; and by the same we have the radius $=\frac{1}{2}\left\{\frac{(18)^2+(12)^2}{12}\right\}=\frac{468}{24}=19\frac{1}{2}$.

By Table XI., arc of 130° to radius 1 = 2·2689280
„ 4° „ = ·0698132
„ 40′ „ = 116355
„ 5′ „ = 14544
„ 30″ „ = 1454
„ 7″ „ = 339
2·3520104
Multiply by 19½
1·1760052
21·1680936
23·520104
Ans. Length of the arc =45·8642028

(4.) Since the radius is 40 and the height 8, the distance of the chord from the centre, or DC in the diagram, is 32; hence by (Trig. 7) cos.$\frac{1}{2}$ arc.$=\frac{32}{40}=\frac{4}{5}=$·800000; therefore half the arc =36° 52′ 11$\frac{1}{2}$″; wherefore the whole angle =73° 44′ 23″.

By Table XI., arc. 70°, to radius 1			=1·2217305
„	3°	„	= ·0523599
„	40′	„	= 116355
„	4′	„	= 11636
„	20″	„	= 970
„	3	„	= 145
			1·2870010
Multiply by radius			= 40
Ans. Length of the arc			=51·4800400

Or, 73° 44′ 23″=$73{\cdot}73969\dot{4}^{\circ}$, which being multiplied by 40 and by ·0174533, gives the product 51·48004005161. Ans.

(5.) 6 feet 9 in. =6·75 feet, and 18 feet 6 in. =18·5 feet, one-half of which is 9·25 feet; therefore the tangent of $\frac{1}{4}$ of the arc. $=\frac{6{\cdot}75}{9{\cdot}25}$, or the Log. tan.$\frac{1}{4}$ arc. =10+Log. 6·75— Log. 9·25=9·863162= the tan. of 36° 7′ 9$\frac{3}{4}$″; hence the whole arc =144° 28′ 39″.

Also the radius $=\frac{(9{\cdot}25)^2+(6{\cdot}75)^2}{2\times 6{\cdot}75}=9{\cdot}713$ very nearly.

By Table XI., arc 140°, to radius 1			= 2·4434610
„	4°	„	= ·0698132
„	20′	„	= 58178
„	8′	„	= 23271
„	30″	„	= 1454
„	9″	„	= 436
			2·5216081
Multiply by radius inverted,			3179
			22·6944729
			1·7651257
			252161
			75648
Ans. Length of the arc			=24·4923795

Problem IX.

(1.) $\left(\frac{333}{2}\times 53\right)\div 9 = 8824{\cdot}5\div 9 = 980{\cdot}5$ sq. yards.

(2.) $\left(\frac{1522}{2}\times 9563\right)\div 144 = 7277443\div 144 = 50537{\cdot}8$ nearly.

(3.) $3{\cdot}1416\times 46\times 46 = 6647{\cdot}6256.$ Ans.
or $\cdot 7854\times 92\times 92 = 6647{\cdot}6256.$ Ans.

(4.) $3{\cdot}1416\times (45)^2\div 9 = 6361{\cdot}74\div 9 = 706{\cdot}86.$ Ans.
or $\cdot 7854\times (90)^2\div 9 = 6361{\cdot}74\div 9 = 706{\cdot}86.$ Ans.

(5.) $\cdot 0795775\times (100)^2 = 795{\cdot}775.$ Ans.

(6.) $\cdot 0795775\times (40)^2 = 127{\cdot}324.$ Ans.

(7.) To find the radius, we have by Note I., Problem VIII., Radius $= \frac{(16)^2+8^2}{2\times 8} = \frac{256+64}{16} = \frac{320}{16} = 20.$

Hence the area $= 3{\cdot}1416\times (20)^2 = 1256{\cdot}64.$ Ans.

(8.) Here the radius is evidently the hypotenuse of a right-angled triangle, of which the sides are half the chord, and its distance from the centre; wherefore

Radius $= \sqrt{(12)^2+5^2} = \sqrt{169} = 13.$
$\therefore$ the area $= 3{\cdot}1416\times 169 = 530{\cdot}9304.$
Or $\cdot 7854\times (26)^2 = 530{\cdot}9304.$ Ans.

Problem X.

(1). $\frac{20{\cdot}5\times 14}{2} = 143{\cdot}5$, or $143\frac{1}{2}$. Ans.

(2.) $\frac{10\times 12}{2} = 60.$ Ans.

(3.) $\frac{100\times 54}{2} = 2700$ sq. feet $= 300$ sq. yards. Ans.

RULE II.

(1.)
$$\begin{array}{lcr}
\text{Here } \cdot0087266\times35 & = & \cdot305431\\
\text{Multiply by } (45)^2 & = & 2025\\ \cline{3-3}
 & & 1527155\\
 & & 610862\\
 & & 610862\\ \cline{3-3}
\text{Ans.} & = & 618\cdot497775 \text{ sq. feet.}
\end{array}$$

(2.) The area of a sector of 30° to radius 1 is
$$\begin{array}{lcr}
=\cdot0087266\times30 & = & \cdot261798\\
\text{Multiply by } (\tfrac{3}{2})^2 & = & 2\tfrac{1}{4}\\ \cline{3-3}
 & & \cdot065449\\
 & & \cdot523596\\ \cline{3-3}
\text{Ans.} & = & \cdot589045 \text{ feet.}
\end{array}$$

Since 30° is the 12th part of 360°, the sector will be the 12th part of the whole circle, and its area may therefore be found as follows:

$$\frac{\cdot7854\times3^2}{12}=\frac{\cdot7854\times3}{4}=\cdot58905 \text{ feet.}$$

(3.) The area of a sector of 90° to radius 1 is
$$\begin{array}{lcr}
=\cdot0087266\times90 & = & \cdot785394\\
\text{Multiply by } 9^2 & = & 81\\ \cline{3-3}
 & & \cdot785394\\
 & & 62\cdot83152\\ \cline{3-3}
\text{Ans.} & = & 63\cdot616914 \text{ sq. feet.}
\end{array}$$

Or, since 90° is the fourth part of 360°, the sector will be the fourth part of the whole circle, and its area may therefore be found as follows:—

$$\frac{\cdot7854\times(18)^2}{4}=\cdot7854\times81=63\cdot6174.$$

(4.) The area of a sector of 60 to radius 1 is
$$\begin{array}{lcr}
=\cdot0087266\times60 & = & \cdot523596\\
\text{Multiply by } (12)^2 & = & 144\\ \cline{3-3}
 & & 2094384\\
 & & 2094384\\
 & & 523596\\ \cline{3-3}
 & & 9)75\cdot397824 \text{ sq. feet.}\\ \cline{3-3}
\text{Ans.} & = & 8\cdot377536 \text{ sq. yards.}
\end{array}$$

Or, since 60° is the sixth part of 360°, the sector will be the sixth part of the whole circle; and since 12 feet = 4 yards, its area may be found as follows:

$$\frac{\cdot 7854\times 64}{6}=\cdot 2618\times 32=8\cdot 3776.$$

(5.) The area of a sector of 72° to radius 1, is

$$\begin{array}{lcr}
=\cdot 0087266\times 72 & = & \cdot 6283152 \\
\text{Multiply by } 5^2 & = & 25 \\
\hline
 & & 31415760 \\
 & & 12566304 \\
\hline
\text{Ans.} & = & 15\cdot 7078800 \text{ inches.}
\end{array}$$

Or, since 72° is the fifth part of 360°, the sector is the fifth part of the whole circle, and therefore $=\frac{3\cdot 1416\times 5^2}{5}=3\cdot 1416\times 5=15\cdot 708$ inches.

(6.) Since the height is 5, and half the chord is 12, the tangent of $\frac{1}{4}$ the arc, (Prob. VIII., Note 1.)$=\frac{5}{12}$; $\therefore$ Log. tan.$\frac{1}{4}$ arc $=10+$ Log. 5— Log. $12=10\cdot 698970-1\cdot 079181=9\cdot 619789=$ tan. 22° 37′ 11·5″, which being multiplied by 4, gives the whole arc = 90° 28′ 46″= $90\cdot 479\dot{4}^\circ$.

Also radius $=\frac{(12)^2+5^2}{2\times 5}=\frac{169}{10}=16\cdot 9$.

The area of a sector of 90° 28′ 46″ to radius 1, is $=\cdot 0087266\times 90\cdot 479\frac{4}{9}=\cdot 7895779$, &c.; which being multiplied by $(16\cdot 9)^2=285\cdot 61$, gives 225·51132.

(7.) Since the radius is 9, and half the chord 3, it is evident that sine of half the arc of the sector $=\frac{3}{9}=\cdot 333333$ = sin. 19° 28′ $16\frac{1}{2}$″, and therefore whole angle of the sector is 38° 56′ 33″=38·9425°; and the area of a sector of 38·9425° to radius 1, is

$$\begin{array}{lcr}
=\cdot 0087266\times 38\cdot 9425 & = & \cdot 339835, \text{ \&c.} \\
\text{Multiply by } 9^2 & = & 81 \\
\hline
 & & 339835 \\
 & & 2718680 \\
\hline
 & & 27\cdot 526635. \quad \text{Ans.}
\end{array}$$

PROBLEM XI.

(1.) Since the radius =20, and the chord 24, the Log. *sine of half* the angle=10+ Log. 12— Log. 20=11·079181 —*1·301030*=9·778151= the sine of 143° 7′ 48·7″, because it is the greater segment that is sought; therefore the whole arc $=286^\circ\ 15'\ 37{\cdot}4''=286{\cdot}2603\dot{8}^\circ$.

Let the number ·0087266 be multiplied by this, so as to retain seven decimal places, as follows:—

$$\begin{array}{lrl}
 & \cdot0087266 & \\
 & \underline{83062{\cdot}682} & \\
 & 17453200 & \\
 & 6981280 & \\
 & 523596 & \\
 & 17453 & \\
 & 5236 & \\
 & 26 & \\
 & \underline{7} & \\
 & 2{\cdot}4980798 & \\
\text{Add} & \underline{\cdot4799995} & =\tfrac{1}{2}\text{ nat. sin. }286^\circ\ 15'\ 37{\cdot}4''. \\
 & 2{\cdot}9780793 & \\
\text{Multiply by} & \underline{400} & =\text{ radius square.} \\
 & 1191{\cdot}2317200 & \text{Ans.}
\end{array}$$

NOTE. The sine of an angle that is greater than 180° is ***minus***, (Trig. 29), which is the reason why the $\frac{1}{2}$ sine of the angle is added in the above exercise. The area might also have been found by taking out the length of the arc of $286^\circ\ 15'\ 37{\cdot}4''$ from Table XI., and multiplying its *half* by the square of the radius, and then adding to the product the area of the triangle, which can easily be found as follows: since the radius is 20, and the chord 24, if the square of half the chord be subtracted from the square of the radius, the remainder will be the square of the perpendicular from the centre upon the chord, or $CE=\sqrt{(20)^2-(12)^2}=\sqrt{256}=16$; hence the area of the triangle $ACB=12\times16=192$.

(2.) Area of a sector of 1° to radius 1, or half the length of an arc of 1° to radius 1 = ·0087266

$$\begin{array}{lcr}
\text{Multiply by the number of degrees} & & \underline{72} \\
 & & 174532 \\
 & & \underline{610862} \\
 & & \cdot6283152 \\
\text{Subtract }\tfrac{1}{2}\text{ sin. }72^\circ & = & \underline{\cdot4755285} \\
 & & \cdot1527867 \\
\text{Multiply by }(12)^2 & = & \underline{144} \\
 & & 6111468 \\
 & & 6111468 \\
 & & \underline{1527867} \\
 & \text{Ans.} & 22{\cdot}0012848
\end{array}$$

(3.) Radius $=\dfrac{(10)^2+4^2}{2\times4}=14\tfrac{1}{2}$, (Problem VIII., Note I.), whence also tan.$\frac{1}{4}$ arc. $= \frac{4}{10}$; $\therefore$ Log. tan.$\frac{1}{4}$ arc. $=10+$ Log. 4— Log. $10=10+\cdot602060-1=9\cdot602060=$ Log. tan. $21^\circ\ 48'\ 5''$; and hence the whole arc $=87^\circ\ 12'\ 20''= 87\cdot20\dot{5}^\circ$; wherefore by Rule II. we have

$$
\begin{array}{lcr}
 & & \cdot0087266 \\
\text{Multiplied by the degrees inverted} & & 5550278 \\
\quad\text{to retain 7 decimals,} & & \overline{6981280} \\
 & & 610862 \\
 & & 17453 \\
 & & 436 \\
 & & 44 \\
 & & 4 \\
 & & \overline{\cdot7610079} \\
\text{Subt. } \tfrac{1}{2}\text{ sin. } 87^\circ\ 12'\ 20'' & = & \cdot499405 \\
 & & \overline{\cdot2616029} \\
\text{Multiply by } (14\tfrac{1}{2})^2 & = & 210\tfrac{1}{4} \\
 & & \overline{\cdot0654007} \\
 & & 26160290 \\
 & & 5232058 \\
\text{Ans.} & & \overline{55\cdot0020097}
\end{array}
$$

(4.) Since 90° is the fourth part of 360°, the sector of 90° is the fourth part of the whole circle; and the triangle being right-angled, its area will be $\frac{1}{2}r^2$; whence the area of the segment will be =

$$
\begin{array}{lcl}
\dfrac{3\cdot1416\times(30)^2}{4} & = & 706\cdot86, \\
\triangle=15\times30 & = & 450, \\
\text{Area of the segment} & = & 256\cdot86.
\end{array}
$$

Or by Rule II.

$$
\begin{array}{lcrl}
 & & \cdot0087266 & \\
\text{Multiply by} & & 90 & \\
 & & \overline{\cdot785394} & \\
\text{Subt. } \tfrac{1}{2}\text{ sin. } 90 & = & \cdot500000 & \\
 & & \overline{\cdot285394} & \\
\text{Multiply by } (30)^2 & = & 900 & \\
 & & \overline{256\cdot8546} & \text{Ans.}
\end{array}
$$

(5.)

$$\begin{array}{lr}
 & \cdot0087266 \\
 & 270 \\ \hline
 & 610862 \\
 & 174532 \\ \hline
 & 2{\cdot}356182 \\
\text{Subt. } \tfrac{1}{2}\text{ sin. } 270^\circ = -\cdot5 & +\ \cdot5 \\ \hline
 & 2{\cdot}856182 \\
\text{Multiply by } (15)^2 & =\quad 225 \\ \hline
 & 14280910 \\
 & 5712364 \\
 & 5712364 \\ \hline
 & 642{\cdot}64095
\end{array}$$

Problem XII.

(1.) Since $(40-30)^2$ is less than $4\times(35)^2$, the centre is within the zone.

OK, or the distance of the centre from the middle of the zone $=\frac{(40+30)(40-30)}{(8\times35)}=\frac{700}{280}=2{\cdot}5$; hence the distance of the centre from the less chord $=\frac{35}{2}+2{\cdot}5=17{\cdot}5+2{\cdot}5=20$.

Again the radius $=\sqrt{(20)^2+(15)^2}=25$.

And $\text{AD}=\sqrt{\tfrac{1}{4}(40-30)^2+(35)^2}=35{\cdot}355337$.

Wherefore the sine of half the angle at the centre subtended by $\text{AD}=\frac{35{\cdot}355337}{50}=\cdot707107=\text{sin. }45^\circ$, and hence the whole angle subtended at the centre by $\text{AD}=90^\circ$.

$$\begin{array}{lr}
 & \cdot0087266 \\
\text{Multiply by} & 90 \\ \hline
 & \cdot785394 \\
\text{Subtract } \tfrac{1}{2}\text{ sin. } 90^\circ = & \cdot5 \\ \hline
 & \cdot285394 \\
\text{Multiply by } (25)^2 = & 625 \\ \hline
 & 1426970 \\
 & 570788 \\
 & 1712364 \\ \hline
 & 178{\cdot}37125\text{, area of one segment.} \\
 & 2 \\ \hline
 & 356{\cdot}7425\text{, area of both segments.} \\
\frac{40+30}{2}\times35=1225, & \text{area of ADCB.} \\ \hline
\text{Ans.} & 1581{\cdot}7425\text{, area of the zone.}
\end{array}$$

(2.) Here $OP=\frac{(32+24)(32-24)-4\times4^2}{8\times4}$

$$=\tfrac{384}{32}=12.$$

$$\text{Also radius }=\sqrt{(16)^2+(12)^2}=\sqrt{400}=20,$$

$$\text{And AD }=\sqrt{\tfrac{1}{4}(32-24)^2+4^2}$$

$$=\sqrt{32}=5{\cdot}656854.$$

Now the sine of half the angle subtended by $AD=\frac{5{\cdot}656854}{40}={\cdot}1414213=\sin.\ 8^\circ\ 7'\ 48''$; therefore the whole angle $=16^\circ\ 15'\ 36''=16{\cdot}26^\circ$.

Next, to find the area of the segment, we have

$$\begin{array}{lr}
 & {\cdot}0087266 \\
\text{Multiplied by} & 16{\cdot}26 \\
\hline
 & 523596 \\
 & 174532 \\
 & 523596 \\
 & 87266 \\
\hline
 & {\cdot}141894516 \\
\tfrac{1}{2}\sin.\ 16^\circ\ 15'\ 36'' = & {\cdot}139998 \\
\hline
 & {\cdot}001896 \\
\text{Multiplied by } R^2 = & 400 \\
\hline
 & {\cdot}7584,\ \text{area of one segment.} \\
 & 2 \\
\hline
 & 1{\cdot}5168,\ \text{area of both segments.} \\
\frac{32+24}{2}\times4 & =112{\cdot} \\
\hline
 & 113{\cdot}5168.\quad \text{Ans.}
\end{array}$$

(3.) Since the chords are equal, they must be on opposite sides of the centre; and since the chords AB and DC are equal, it is plain that AD and CB are each equal to QP; but QP is =AB or DC, therefore the figure ABCD is a square, and its area will evidently be 2500.

Again, the angle at the centre will evidently be 90°, and the radius is obviously $=\sqrt{(25)^2+(25)^2}=\sqrt{1250}=25\sqrt{2}$.

Hence the area of one of the segments will be

		·0087266	
Multiply by		90	
		·785394	
Subt. $\frac{1}{2}$ sin. 90°	=	·5	
		·285394	
Multiply by R²	=	1250	
		14269700	
		570788	
		285394	
		356·7425,	area of one segment.
		2	
		713·485,	area of both segments.
Add area of square		2500	
		3213·485.	Ans.

(4.) Since the radius is 20 and the less chord 20, the sine of half the angle subtended at the centre by the less chord $=\frac{10}{20}=\frac{1}{2}=$ sin. 30°, (Trig. 84), and therefore the whole angle $=60°$; hence the area of the segment cut off by the less chord is

		·0087266
Multiply by		60
		·523596
Subtract $\frac{1}{2}$ sin. 60°	=	·433013
		·090583
Multiply by R²	=	400
		36·2332

And the area of the semicircle $=·7854\times\frac{(40)^2}{2}$

·7854 × 800	=628·32
Subt. area of segment	=36·2332
	592·0868. Ans.

PROBLEM XIII.

(1.) $(5+4)(5-4)\times·7854=9\times·7854=7·0686$. Ans.

(2.) $(16+10)(16-10)\times·7854=26\times6\times·7854=156\times·7854=122·5224$. Ans.

(3.) $(24+18)(24-18)\times{\cdot}7854=42\times6\times{\cdot}7854=252\times{\cdot}7854=197{\cdot}9208$. Ans.

(4.) $(15+12)(15-12)\times{\cdot}7854=27\times3\times{\cdot}7854=81\times{\cdot}7854=63{\cdot}6174$. Ans.

PROBLEM XIV.

(1.) By Problem XI., the tangent of $\frac{1}{4}$ of the degrees in the arc of the greater segment $=\frac{9}{18}=\frac{1}{2}$; $\therefore$ Log. tan. $\frac{1}{4}$ arc. $=10+$ Log. $1-$ Log. $2=10-{\cdot}301030=9{\cdot}698970=$ Log. tan. $26^\circ\ 33'\ 54''$; therefore the degrees in the arc $=106^\circ\ 15'\ 36''=106{\cdot}26^\circ$.

Also the radius of the circle $=\frac{(18)^2+9^2}{2\times9}=22{\cdot}5$; hence the area of the greater segment will be found as follows:

		·0087266
Multiplied by 106·26°		62601 inverted.
		8726600
		523596
		17453
		5236
		·9272885
Subtract $\frac{1}{2}$ sin. $106^\circ\ 15'\ 36''$	=	·4800005
		·447288
Multiply by R^2	=	$506\frac{1}{4}$
		·111822
		2683728
		2236440
		226·439550, area of the

greater segment.

Again, the tangent of $\frac{1}{4}$ of the degrees in the arc of the less segment $=\frac{6}{18}=\frac{1}{3}$; hence Log. tan. $\frac{1}{4}$ arc $=10+$ Log. $1-$ Log. $3=10-{\cdot}477121=9{\cdot}522879=$ Log. tan. $18^\circ\ 26'\ 6''$; therefore the whole arc $=73^\circ\ 44'\ 24''=73{\cdot}74^\circ$.

Also the radius of the circle $=\frac{(18)^2+6^2}{2\times6}=30$; hence the *area of the* segment will be found as follows:

		·0087266
Multiplied by 73·74		4737 inverted.
		6108620
		261798
		61086
		3490
		·6434994
Subt. ½ sin. 73° 44′ 24″	=	·4800005
		·1634989
Multiply by R^2	=	900
		147·1490100, area of the

less segment.

Greater segment	=	226·43955
Less do.	=	147·14901
Difference,	=	79·29054
Sum,	=	373·58856

(2.) The tangent of $\frac{1}{4}$ of the degrees in the greater arc $=\frac{5}{15}=\frac{1}{3}$; hence Log. tan. $\frac{1}{4}$ arc$=10+$ Log. 1 — Log. 3 $=10-\cdot477121=9\cdot522879=$ Log. tan. 18° 26′ 6″; therefore the whole arc $=73°\ 44′\ 24″=73\cdot74°$.

Also the radius of the circle $=\frac{(15)^2+5^2}{2\times5}=25$; hence the area of the greater segment will be found as follows:

		·0087266
Multiplied by 73·74		4737 inverted.
		6108620
		261798
		61086
		3490
		·6434994
Subtract ½ sin. 73° 44′ 24″		·4800005
		·1634989
Multiply by $(25)^2$	=	625
		8174945
		3269978
		9809934
		102·1868125, area of the

greater segment.

Again, the tangent of $\frac{1}{4}$ of the degrees in the arc of the less segment $=\frac{5}{15}=\frac{1}{5}=$ tan. 11° 18′ $35\frac{3}{4}$″, and therefore the whole arc $=45°\ 14′\ 23″=45\cdot2397\dot{2}°$; hence the area *of the segment may* be found as follows:

		·0087266
Multiply by $45{\cdot}2397\dot{2}^\circ$		2793254 inverted.
		3490640
		436330
		17453
		2618
		785
		61
		2
		·3947889
Subtract $\frac{1}{2}$ sin. 45° 14′ 23″	=	·3550295
		·0397594
Multiply by $R^2=(39)^2$	=	1521
		397594
		795188
		1987970
		397594
Less segment	=	60·4740474
Greater segment	=	102·1868125
Difference	=	41·7127651
Sum	=	162·6608599

PROBLEM XV.

(1.)		Odd.	Even.
	20	43	31
	90	68	57
	110	111	78
	222	2	166
	664	222	4
Sum =	996		664
$\frac{60}{5}$ =	20		
	19920 Ans.		

(2.)		Odd.	Even.
	112	93	110
	10	71	80
	122	52	60
	492	30	41
	1236	246	18
Sum =	1850	2	309
		492	4
			1236

$1850 \times \frac{19{\cdot}6}{3} = 12086\frac{2}{3}$. Ans.

		Odd.	Even.
(3.)	0	38	20
	78	61	50
	$\overline{78}$	$\overline{99}$	70
	198	2	$\overline{140}$
	560	$\overline{198}$	4
Sum,	$\overline{836}$		$\overline{560}$

$$836\times\frac{61}{3}=16998\tfrac{2}{3}.\quad\text{Ans.}$$

MENSURATION OF SOLIDS.

(1.) $\text{Surface}=\dfrac{16\times16\times6}{144}=10\tfrac{2}{3}$ feet.

$\text{Solidity}=\dfrac{16\times16\times16}{1728}=2\tfrac{10}{27}$ feet.

(2.) $6\times6\times6=216$, surface and solidity.

(3.) $(2+1\frac{1}{2})\times2=7$, the perimeter, which being multiplied by the length 12, gives 84, the surface of the sides.

$$\begin{array}{lcl} \text{To which add } 2\times1\frac{1}{2}\times2 & = & \underline{\ \ 6} \\ & & 90,\ \text{surface.} \\ 2\times1\frac{1}{2}\times12 & = & 36,\ \text{solidity.} \end{array}$$

(4.)
$$\begin{array}{lcrl} 1\frac{3}{4}\times4 & = & 7, & \text{perimeter.} \\ 7\times16 & = & 112, & \text{surface of the sides.} \\ 1\frac{3}{4}\times1\frac{3}{4}\times2 & = & \underline{6\frac{1}{8}}, & \text{surface of the ends.} \\ & & 118\frac{1}{8}, & \text{whole surface.} \\ 1\frac{3}{4}\times1\frac{3}{4}\times16 & = & 49, & \text{the solidity.} \end{array}$$

(5.)
$$\begin{array}{lcrl} \text{Tabular area} & = & \cdot433013, & \text{page 67.} \\ \text{Multiply by } 2^2 & = & \underline{4} & \\ & & 1\cdot732052 & \\ & & \underline{2} & \\ & & 3\cdot464104, & \text{surface of the ends.} \\ 12\frac{1}{2}\times2\times3 & = & \underline{75\cdot\qquad} & \text{surface of the sides.} \\ & & 78\cdot464104, & \text{whole surface.} \\ 1\cdot732052\times12\frac{1}{2} & = & 21\cdot65065, & \text{solidity.} \end{array}$$

(6.) Tabular area = 1·720477, page 67.
Mult. by $(3\frac{1}{2})^2$ = $12\frac{1}{4}$
430119
20·645724
21·075843
2
42·151686, surface of the ends.
$3\frac{1}{2} \times 5 \times 15\frac{1}{3}$ = 268·333333, surface of the sides.
310·485019, whole surface.
Area of an end = 21·075843
Multiply by $15\frac{1}{3}$ $15\frac{1}{3}$
7·025281
105379215
21075843
323·162926, solidity.

(7.) Tabular area = 2·598076, page 67.
Multiply by $(11)^2$ = 121
2·598076
5196152
2598076
314·367196, area of one end.
2
144)628·734392, area of both ends.
4·366211 sq. feet.
$\frac{11 \times 6 \times 3\frac{3}{4}}{12}$ = 20·625 area of sides.
24·991211, whole surface.
Area of an end 2·183105
Mult. by the height = $3\frac{3}{4}$
4|6·549315
1·637329
6·549315
8·186644, solidity.

(8.) Tabular area 4·828427, page 67.
Multiply by $(\frac{1}{2})^2$ = $\frac{1}{4}$
1·207107, area of an end.
2
2·414214, area of both ends.
$\frac{1}{2} \times 8 \times 14$ = 56 area of the sides.
58·414214, whole surface.
Area of an end as above 1·207107
Multiply by the height = 14
16·899498, solidity.

PROBLEM II.

(1.) $\cdot 7854 \times (3^2 = 9) \quad = 7\cdot0686$, area of the end.

$$\begin{array}{r} 7\cdot0686 \\ 2 \\ \hline 14\cdot1372 \end{array}$$

$14\cdot1372$, area of both ends.

$3\cdot1416 \times 3 \times 7\frac{1}{2} \quad = 70\cdot6860$, curved surface.

$84\cdot8232$, whole surface.

Area of an end as above $7\cdot0686$
Multiply by the height, $7\frac{1}{2}$

$$\begin{array}{r} 7\cdot0686 \\ 7\frac{1}{2} \\ \hline 3\cdot5343 \\ 49\cdot4802 \\ \hline 53\cdot0145 \end{array}$$

$53\cdot0145$, solidity.

(2.) $3\cdot1416 \times 5 \times 30 \quad = 471\cdot24$ feet, convex surf.
$\cdot 7854 \times 5 \times 5 \quad = 19\cdot635$, area of the base.
Mult. by the height $= 30$

$589\cdot05$ cubic feet, solidity.

(3.) $\cdot 7854 \times 4 \times 4 \quad = 12\cdot5664$, area of the base.
Mult. by depth $= 36$

$$\begin{array}{r} 75\cdot3984 \\ 376992 \\ \hline 27\left\{\begin{array}{r} 9)\overline{452\cdot3904} \\ 3)\overline{50\cdot2656} \end{array}\right. \end{array}$$

$16\cdot7552$ cubic yards.

$$\begin{array}{rl|l|r} 10s. & = & \frac{1}{2} & 16\cdot7552 \\ 6d. & = & \frac{1}{20} & \cdot8\,3776 \\ & & & \cdot41888 \\ \hline \end{array}$$

L. $8\cdot79648$ = L.8, 15s. 11d. Ans.

(4.) $\cdot 7854 \times 1\frac{3}{4} \times 1\frac{3}{4} \times 5\frac{1}{4} = 12\cdot627758$, solidity.
$3\cdot1416 \times 1\frac{3}{4} \times 5\frac{1}{4} \quad = 28\cdot86347$ sq. feet, convex. surf.
4 ac. 2 ro. 20 poles $= 201465$ sq. feet.
$\therefore \frac{201465}{28\cdot86347} \quad = 6979\cdot93$, number of revolutions.

PROBLEM III.

(1.) $30 \times 4 \quad = 120$, the perimeter.
$120 \times \frac{25}{2} \quad = 1500$, slant surface.
$30 \times 30 \quad = 900$, area of the base.
Sum $= 2400$, whole surface.

$\sqrt{(25)^2 - (15)^2} = \sqrt{400} = 20$, perp. altitude.
$\therefore 900 \times \frac{20}{3} \quad = 6000$, solidity.

(2.) By Note I., the perpendicular from the centre on the middle of a side $=\frac{1}{2}S\cot.\frac{180}{n}=20\cot.30^\circ$.

$$\begin{array}{lrl}
\text{Log. } 20 & = & 1{\cdot}301030 \\
\text{Log. cot. } 30^\circ & = & \underline{10{\cdot}238561} \\
 & & 1{\cdot}539591 \\
 & & \underline{\qquad\qquad 2} \\
1200= & & 3{\cdot}079182, \text{ Log. of square.}
\end{array}$$

$(\text{Perp.})^2=3600$

Sum $=\overline{4800}$, the square of the slant height.

$\therefore$ the slant height $=\sqrt{4800}=69{\cdot}282032$.

Hence $\frac{240\times69{\cdot}282032}{2}=8313{\cdot}8438$, slant surface.

$2{\cdot}598076\times40\times40=\underline{4156{\cdot}9216}$, area of the base.

$\therefore 12470{\cdot}7654=$ whole surface.

$$\begin{array}{lrl}
 & & 4156{\cdot}9216, \text{ area of base.} \\
\text{Mult. by } \frac{60}{3} & = & \underline{\qquad\qquad 20} \\
 & & 83138{\cdot}432, \text{ solidity.}
\end{array}$$

(3.) By Note I., the perpendicular from the centre on the middle of the side $=\frac{1}{2}\cdot20\cot.60^\circ=\frac{10}{\sqrt{3}}$.

$$\begin{array}{lrl}
\therefore (\text{perp.})^2= & \frac{100}{3}= & 33{\cdot}3333 \\
\text{Square of the height} & = & \underline{400} \\
 & & 433{\cdot}3333, \&c. = (\text{slant height})^2. \\
\therefore 20{\cdot}81666 & = & \text{the slant height.} \\
\text{hence } \frac{60\times20{\cdot}81666}{2} & = & 624{\cdot}4998, \text{ slant surface.} \\
{\cdot}433013\times(20)^2 & = & \underline{173{\cdot}2052}, \text{ area of the base.} \\
 & & 797{\cdot}705, \text{ whole surface.}
\end{array}$$

$$\begin{array}{lrl}
\text{Area of the base} & = & 173{\cdot}2052 \\
\text{Multiply by } \frac{20}{3} & = & \underline{\qquad\qquad 6\frac{2}{3}} \\
 & & 3\,|\,\underline{346{\cdot}4104} \\
 & & 115{\cdot}4701 \\
 & & \underline{1039{\cdot}2312} \\
 & & 1154{\cdot}7013, \text{ solidity.}
\end{array}$$

(4.) $\frac{2\times5\times4}{2}$ $=20$, slant surface.
$1·720477\times2^2=$ 6·881908, area of the base.
26·881908, whole surface.
Perpendicular from the centre on the side $=$
$\frac{1}{2}\times2$ cot. 36° $=$ cot, 36°.
∴ Log. perp. $=$ Log.cot. 36°$-10=$ ·138739.
2
∴ (Perp.)^2 $=$ 1·89443 $=$ ·277478.
Square of slant height $=$ 16·
Difference $=$ 14·10557, sq. of altitude.
Hence altitude $=\sqrt{14·10557}=$ 3·755738.
Whence $6·881908\times\frac{3·755736}{3}=$ 8·61555, solidity.

(5.) Tabular area $=$ 4·828427, page 67.
Multiply by $(3\frac{1}{4})^2$ $=$ $10\frac{9}{16}$
16|43·455843
2·715990
48·284270
51·00026
Multiply by $\frac{15}{3}$ $=$ 5
255·00130, solidity.

Problem IV.

(1.) $·7854\times7\times7$ $=$ 38·4846, area of the base.
Multiply by $\frac{10}{3}$ $=$ $3\frac{1}{3}$
12·8282
115·4538
128·282, solidity.

(2.) $3·1416\times15\times15$ $=$ 706·86, area of the base.
$3·1416\times30\times\frac{25}{2}$ $=$ 1178·1, convex surface.
1884·96, whole surface.
$\sqrt{(25)^2-(15)^2}=\sqrt{400}=$ 20, perp. height.
Area of the base $=$ 706·86
Multiply by $\frac{20}{3}$ $=$ $6\frac{2}{3}$
3|1413·72
471·24
4241·16
4712·4, solidity.

(3.) $\sqrt{(12)^2+5^2}=\sqrt{169}$ $=13$, the slant height.
$3{\cdot}1416\times24\times\frac{13}{2}$ $=490{\cdot}0896$, convex surf.
$\frac{{\cdot}7854\times24\times24\times5}{3}$ $=753{\cdot}984$, solidity.

(4.) $\frac{94{\cdot}248}{3{\cdot}1416}$ $=30$, the diameter of the base.
$\sqrt{(36)^2+(15)^2}=\sqrt{1521}=39$, the slant height.
$\frac{94{\cdot}246\times39}{2}$ $=$ $1837{\cdot}797$, convex surface.
${\cdot}7854\times30\times30=$ $706{\cdot}86$, area of the base.
$2544{\cdot}657$, whole surface.
Area of the base, $706{\cdot}86$
Multiply by $\frac{36}{3}=$ 12
$8482{\cdot}32$, solidity.

PROBLEM V.

(1.) $\frac{1}{2}(15-9)\text{ cot. }\frac{180^\circ}{3}=3\text{ cot. }60^\circ=3\times\frac{1}{\sqrt{3}}=\sqrt{3}$

$=$ the difference of the perpendiculars from the centres of the ends upon the sides, the square of which increased by the square of the altitude, will be the square of the slant height.

$\therefore\sqrt{(12)^2+3}=\sqrt{147}=12{\cdot}124356=$ the slant height.
Hence $\frac{3(15+9)}{2}\times12{\cdot}124356=436{\cdot}476816$, surf. of sides.
$\{(15)^2+9^2\}\times{\cdot}433013$ $=132{\cdot}501978$, area of ends.
$568{\cdot}978794$, whole surf.
$\{(15)^2+15\times9+9^2\}\times\frac{12}{3}\times{\cdot}433013=763{\cdot}834932$, solidity.

(2.) $\sqrt{(12)^2+(1\frac{1}{2})^2}=12{\cdot}093382$, slant height.
$\frac{4(5+2)}{2}\times12{\cdot}093382=169{\cdot}307348$, slant. surf.
5^2+2^2 $=29$, area of the ends.
$198{\cdot}307348$, whole surfaces.
5^2 $=25$
2^2 $=4$
5×2 $=10$
$39\times\frac{12}{3}=156$, solidity.

(3.) $(2\frac{1}{3})^2 = 5\frac{4}{9} = 5\frac{16}{36}$
$(1\frac{1}{6})^2 = 1\frac{13}{36} = 1\frac{13}{36}$
$2\frac{1}{3} \times 1\frac{1}{6} = 2\frac{13}{18} = 2\frac{26}{36}$

$9\frac{19}{36} \times \frac{18\frac{3}{4}}{3} = \dot{5}9{\cdot}5486\dot{1}$, solid.

(4.) $\frac{1}{2}(4-2)\text{ cot. }\frac{180^\circ}{6} = \text{cot. }30^\circ = \sqrt{3}$, the difference of the perpendiculars from the centres of the ends upon the sides; hence

$\sqrt{9^2+3} = \sqrt{84} = 9{\cdot}165151$, slant side.
Mult. by $\frac{1}{2}(4+2)\times 6$ $\quad 18$ semiperimeters.

$73{\cdot}321208$
9165151
$164{\cdot}972718$, slant surface.

Tab. area of hex. $= 2{\cdot}598076$
Mult. by $(4^2+2^2) = 20$
$51{\cdot}961520$, area of ends.
$\therefore 216{\cdot}934238 =$ whole surf.

$4^2 = 16$
$2^2 = 4$
$4\times 2 = 8$
$\overline{28} \times 2{\cdot}598076 \times \frac{9}{3} = 218{\cdot}238384$, solidity.

(5.) $\sqrt{4^2+1^2} = \sqrt{17} = 4{\cdot}123106$, slant height.
$\frac{1}{2}(5+3)\times 3{\cdot}1416 \times 4{\cdot}123106 = 51{\cdot}812599$, slant. surf.
$\cdot 7854 \times (5^2+3^2) = 26{\cdot}7036$, ends.
$78{\cdot}516199$, whole surf.

$5^2 = 25$
$3^2 = 9$
$5\times 3 = 15$
$\overline{49} \times \cdot 7854 \times \frac{4}{3} = 51{\cdot}3128$, solidity.

(6.) $\frac{1}{2}\times 3{\cdot}1416 \times (20+10) \times 13 = 612{\cdot}612$, slant surf.
$\cdot 7854 \times (20^2+10^2) = 392{\cdot}7$, ends.
$1005{\cdot}312$, whole surf.

Perp. height $= \sqrt{13^2-5^2} = \sqrt{144} = 12$.
$20^2 = 400$
$10^2 = 100$
$20\times 10 = 200$
$\overline{700} \times \cdot 7854 \times \frac{12}{3} = 2199{\cdot}12$, solidity.

PROBLEM VI.

(1.) $(35\times2+55)\times17{\cdot}145\times\frac{15}{6}=5357{\cdot}8125$, solid in.

And $\frac{5357{\cdot}8125}{1728}=3{\cdot}1006$ solid feet.

(2.) $(9\times2+11)\times10\times\frac{4}{6}=193\frac{1}{3}$ cubic inches.

(3.) $(26\times2+15)\times28\times\frac{18}{6}=5628$ solid inches.

And $\frac{5628}{1728}=3$ feet 444 inches.

PROBLEM VII.

(1.)
$$\begin{aligned}14\times12&=168\\6\times4&=24\\20\times16&=320\\&\overline{512}\end{aligned}$$
$\frac{512}{144}\times\frac{30}{6}=17\frac{7}{9}$ solid feet.

(2.)
$$\begin{aligned}28\times18&=504\\16\times10&=160\\44\times28&=1232\\&\overline{1896}\end{aligned}$$
$\frac{1896}{144}\times\frac{9}{6}=19\frac{3}{4}$ solid feet.

(3.)
$$\begin{aligned}32\times20&=640\\16\times10&=160\\48\times30&=1440\\&\overline{2240}\end{aligned}$$
$\frac{2240}{144}\times\frac{24}{6}=62\frac{2}{9}$ solid feet.

(4.)
$$\begin{aligned}80\times56&=4480\\42\times30&=1260\\122\times86&=10492\\&\overline{16232}\end{aligned}$$
$\frac{16232\times48\div6}{1728}=75\frac{4}{27}$ solid feet.

PROBLEM VIII.

(1.) The height $=7912\times$ sin. 23° 28′.

The circumference $=7912\times3{\cdot}1416$, and the product of these two will be the surface sought, which is therefore $=(7912)^2\times3{\cdot}1416\times$ sin. 23° 28′; hence

2 Log. 7912 = 7·796572
Log. 3·1416 = ·497151
Log. sin. 23° 28′ = 9·600118
∴ Ans. 78314300 = 7·893841

(2.) 7912÷2 = 3956, radius of the sphere.
3956 — 3956 cos. 23° 28′ = the height of the zone.
7912 × 3·1416 = circumference of the sphere.
Log. 3956 = 3·597256
Log. cos. 23° 28′ = 9·962508
3628·808 = 3·559764 = Log. 3956 cos. 23° 28′.
∴ 3956 -- 3628·808 = 327·192, height.
Log. 327·192 = 2·514803
Log. 7912 = 3·898286
Log. 3·1416 = ·497151
∴ Ans. = 8132810 = 6·910240

(3.) The height of the temperate zone is = (cos. 23° 28′— sin. 23° 28′) × 3956, which being multiplied by 7912 × 3·1416, will give the surface required.
cos. 23° 28′ = ·917292
sin. 23° 28′ = ·398215
Difference = ·519077, hence by logarithms we have the surface as follows:—
Log. ·519077 = $\bar{1}$·715231
Log. 3956 = 3·597256
Log. 7912 = 3·898286
Log. 3·1416 = ·497151
∴ Ans. = 51041500 = 7·707924

(4.) The height of the whole sphere is evidently 7912, and hence 7912 × 7912 × 3·1416 = the whole surface; which being wrought by logarithms, is as follows:
2 Log. 7912 = 7·796572
Log. 3.1416 = ·497151
∴ Ans = 196663000 = 8·293723

PROBLEM IX.

(1.) ·5236 × 3 × 3 × 3 = 14·1372 cubic feet.

(2.) ·5236 × 4 × 4 × 4 = 33·5104 cubic inches.

(3.) By Logarithms.

$$\begin{array}{lr} 3\text{ Log. }7912 & = 11{\cdot}694858 \\ \text{Log. }{\cdot}5236 & = \bar{1}{\cdot}719000 \\ \therefore \text{Ans.} = 259333000000 = & \overline{11{\cdot}413858} \end{array}$$

(4.)

$$\begin{array}{lr} 3\text{ Log. }16 & = 3.612360 \\ \text{Log. }{\cdot}5236 & = \bar{1}{\cdot}719000 \\ \therefore \text{Ans.} = 2144{\cdot}66 & = \overline{3{\cdot}331360} \end{array}$$

Or by Numbers.

$${\cdot}5236 \times 16 \times 16 \times 16 = 2144{\cdot}6656. \quad \textbf{Ans.}$$

PROBLEM X.

(1.) By Rule II.

$(20 \times 3 - 9 \times 2) \times 9^2 \times {\cdot}5236 = 1781{\cdot}2872$, solidity.

(2.) By Rule I.

$(3 \times 10^2 + 9^2) \times 9 \times {\cdot}5236 = 1795{\cdot}4244$, solidity.

(3.) By Rule I.

$(3 \times 16^2 + 8^2) \times 8 \times {\cdot}5236 = 3485{\cdot}0816$, solidity.

(4.) By Rule II.

$(3 \times 30 - 2 \times 24) \times 24^2 \times {\cdot}5236 = 12666{\cdot}9312$, sol.

PROBLEM XI.

(1.)

$$\begin{array}{ll} 2^2 & = 4 \\ (1\tfrac{1}{2})^2 & = 2\tfrac{1}{4} \\ \text{Sum,} & \quad \overline{6\tfrac{1}{4}} \\ \text{Mult. by} & \quad 3 \\ & \quad \overline{18\tfrac{3}{4}} \\ +(2\tfrac{1}{2})^2 & = 6\tfrac{1}{4} \\ & \quad \overline{25} \times 2\tfrac{1}{2} \times {\cdot}5236 = 32{\cdot}725 \text{ solidity.} \end{array}$$

(2.)

$$\begin{array}{r} (20)^2 = 400 \\ (24)^2 = 576 \\ \overline{976} \\ \text{Mult. by} \quad 3 \\ \overline{2928} \\ +(8)^2 = \quad 64 \\ \overline{2992} \times 8 \times {\cdot}5236 = 12532{\cdot}8896. \quad \text{Ans.} \end{array}$$

$$\begin{array}{ll}
(3.) & 3^2\times 6=54 \\
 & +4^2\ =16 \\
 & \qquad\overline{70}\times 4\times \cdot 5236=146\cdot 608, \text{ solidity.}
\end{array}$$

$$\begin{array}{llr}
(4.) & 4^2 & =16 \\
 & 4^2 & =16 \\
 & & \overline{32} \\
 & \text{Mult. by} & 3 \\
 & & \overline{96} \\
 & +8^2 & =64 \\
 & & \overline{160}\times 8\times \cdot 5236=670\cdot 208, \text{ solidity.}
\end{array}$$

Problem XII.

(1.) By Problem XI., Mensuration of Surfaces, find the area of the generating segment as follows:

$\frac{2^2+(1\frac{1}{2})^2}{2\times 1\frac{1}{2}}=\frac{6\frac{1}{4}}{3}=2\frac{1}{12}$ feet, the radius.

Tan. $\frac{1}{4}$ arc $=\frac{1\frac{1}{2}}{2}=\frac{3}{4}$; $\therefore$ Log. tan. $\frac{1}{4}$ arc $=10+$ Log. 3 $-$ Log. 4 $=10+\cdot 477121-\cdot 602060=9\cdot 875061$; $\therefore \frac{1}{4}$ arc $=36°\ 52'\ 11\frac{1}{2}''$; hence the whole arc $=$ $147°\ 28'\ 46''=147\cdot 479\dot{4}°$; wherefore

$$\begin{array}{lrl}
 & \cdot 0087266 & \\
\text{Multiply by} & \underline{44974\cdot 741} & \text{inverted.} \\
 & 8726600 & \\
 & 3490640 & \\
 & 610862 & \\
 & 34907 & \\
 & 6108 & \\
 & 785 & \\
 & 35 & \\
 & 3 & \\
 & \overline{1\cdot 2869940} & \frac{1}{2}\text{ arc to } R=1. \\
\text{Subtract } \frac{1}{2}\text{ sin. } 147°\,28'\,46'' \ = & \cdot 268801 & \\
 & \overline{1\cdot 018193} & \\
\text{Mult. by } (2\frac{1}{12})^2 & \underline{2\frac{1}{12}} & \\
 & 84849 & \\
 & 2\cdot 036386 & \\
 & \overline{2\cdot 121235} & \\
 & \underline{2\frac{1}{12}} & \\
 & 176769 & \\
 & 4242470 & \\
\textit{Carried over,} & \overline{4\cdot 419239} &
\end{array}$$

Carried over, 4·419239

Mult. by $\frac{7}{24}$ $\frac{7}{24}$

24)30·934673

1·288945 subtract from.

$\frac{1}{3}(2)^3$ = 2·666666

1·377721

Mult. by 4665·21 inverted.

13·777210

2·755442

688860

82663

8266

551

Ans. 17·312992, solid. in feet

Again, to find the surface, we have as above,

$\frac{1}{2}$ arc to radius 1 = 1·286994

2

Whole arc to radius 1 2·573988

Mult. by $R = 2\frac{1}{12}$ $2\frac{1}{12}$

214499

5·147976

5·362475

Mult. by central dist. $\frac{7}{12}$

12)37·537325

3·128110 subtract from

$4 \times 2\frac{1}{12}$ = 8·333333

5·205223

Mult. by 6·2832 2382·6 inverted.

31·231338

1·041045

416418

15616

1041

Ans. 32·705458 surface.

(2.) $\frac{(7\frac{1}{2})^2+4^2}{2\times 4} = \frac{72\frac{1}{4}}{8} = \frac{289}{32} = 9\frac{1}{32}$, the radius.

$Tan. \frac{1}{4}$ arc $= \frac{4}{7\frac{1}{2}} = \frac{8}{15}$; hence Log. tan. $\frac{1}{4}$ arc = 10

Log. 8 — Log. 15 = 10 + ·903090 — 1·176091 = 9·726999 = Log. tan. 28° 4′ 21″; therefore the whole arc = 112° 17′ 24″ = 112·29°; hence,

$$
\begin{array}{lr}
 & \cdot0087266 \\
\text{Mult. by } 112\cdot29 \quad = & \underline{92\cdot211} \text{ inverted.} \\
 & \cdot8726600 \\
 & 872660 \\
 & 174532 \\
 & 17453 \\
 & \underline{7854} \\
 & \cdot9799099 \\
\text{Subt. } \tfrac{1}{2} \text{ sin. } 112^\circ\ 17'\ 24'' \quad = & \underline{\cdot462638} \\
 & \cdot5172719 \\
\text{Mult. by } (9\tfrac{1}{32})^2 \quad = & \underline{5674365\cdot18} \text{ inverted.} \\
 & 41\cdot3817520 \\
 & 5172719 \\
 & 2586360 \\
 & 310363 \\
 & 15518 \\
 & 2069 \\
 & 362 \\
 & 31 \\
 & \underline{3} \\
 & 42\cdot1904945 \\
\text{Mult. by} & \underline{2\tfrac{33}{64}} \\
 & 84\cdot3809890 \\
\tfrac{32}{64} = \tfrac{1}{2} & 21\cdot0952472 \\
\tfrac{1}{64} = \tfrac{1}{32} \text{ of } \tfrac{32}{64} & \underline{6592265} \\
 & 106\cdot1354627 \text{ subtract from.} \\
\dfrac{(7\cdot5)^3}{3} \quad = & \underline{140\cdot625} \\
 & 34\cdot4895373
\end{array}
$$

And 34·4895373 being multiplied by 12·5664,
Gives 433·40932 inches, the solidity.

(3.) $\dfrac{10^2+(7\frac{1}{2})^2}{2\times7\frac{1}{2}} = \dfrac{156\frac{1}{4}}{15} = \dfrac{625}{60} = 10\tfrac{5}{12}$, the radius.

Tan. $\frac{1}{4}$ arc $= \dfrac{7\frac{1}{2}}{10} = \frac{15}{20} = \frac{3}{4}$; ∴ Log. tan. $\frac{1}{4}$ arc = 10 + Log. 3 — Log. 4 = 9·875061 = Log. tan. 36° 52′ 11½″; hence the whole arc = 147° 28′ 46″ = 147·479$\dot{4}$°; wherefore,

·0087266

Multiplied by 147·4794° 44974741 inverted.

Gives 1·286994 ½ arc to R=1.

Subtract ½ sin. 147° 28′ 46″ = 268801

1·018193

Mult. by $(10\frac{5}{12})^2$ = 449605·801 inverted.

101819300
8145544
509097
6109
916
41
4

110·481011

Mult. by $\frac{1}{2}(10\frac{5}{12}-7\frac{1}{2})$ = $1\frac{11}{24}$

110·481011
50·637130

161·118141 subt. from.

$\frac{1}{3}\,10^3$ 333·333333

172·215192

Mult. by 12·5664 466521 inverted.

1722151920
344430384
86107596
10332912
1033291
68886

Ans. 2164·124989 solidity.

Again, to find the surface, we have as above,

½ arc to radius 1 = 1·286994
2

2·573988

Mult. by radius = $10\frac{5}{12}$

12)12·869940

1·072495
25·739880

26·812375

Mult. by $(10\frac{5}{12}-7\frac{1}{2})$ = $2\frac{11}{12}$

12)294·936125

24·578010

$$\begin{array}{lrl}
\text{Brought over,} & 24{\cdot}578010 & \\
 & 53{\cdot}624750 & \\ \cline{2-2}
 & 78{\cdot}202760 & \text{subt. from.} \\
20\times 10\tfrac{5}{12} & =\ 208{\cdot}333333 & \\ \cline{2-2}
 & 130{\cdot}130573 & \\
\text{Mult. by } 6{\cdot}2832 & 2382{\cdot}6 & \text{inverted.} \\ \cline{2-2}
 & 780{\cdot}783438 & \\
 & 26{\cdot}026115 & \\
 & 10{\cdot}410446 & \\
 & 390392 & \\
 & 26026 & \\ \cline{2-2}
\text{Ans.} & 817{\cdot}636417 & \text{surface.}
\end{array}$$

PROBLEM XIII.

(1.) $or=40$, $mn=32$, $AD=24$,

$$\text{Radius} = \frac{Ds^2+Sn^2}{2Sn} = \frac{20^2+4^2}{2\times 4} = \frac{416}{8} = 52.$$

$\therefore Ov=52-16=36$, central distance.

$\therefore Ev^2=52^2-36^2=88\times 16=1408$,

$$\frac{Ds}{DO}=\tfrac{20}{52}={\cdot}384615 \text{ nat. sin. } DOn=22^\circ\ 37'\ 11\tfrac{1}{2}''$$

$$\begin{array}{r}
2 \\ \hline
\therefore \angle DOC=45^\circ\ 14'\ 23'' \\
=45{\cdot}2397\dot{2}^\circ
\end{array}$$

The area of the segment DCn will therefore be found as follows, by Problem XI., Mens. of Surf.

$$\begin{array}{lrl}
 & {\cdot}0087266 & \\
\text{Multiply by } 45{\cdot}2397\dot{2} & 2793254 & \text{inverted.} \\ \cline{2-2}
 & 3490640 & \\
 & 436330 & \\
 & 17453 & \\
 & 2618 & \\
 & 785 & \\
 & 61 & \\
 & 2 & \\ \cline{2-2}
 & {\cdot}3947889 & \\
\text{Subt. } \tfrac{1}{2} \text{ nat. sin. } 45^\circ\ 14'\ 23'' & =\ {\cdot}3550295 & \\ \cline{2-2}
 & {\cdot}0397594 & (1.) \\
\text{Mult. by } R^2=(52)^2 & =\ 2704 & \\ \cline{2-2}
\textit{Carried over,} & 1590376 &
\end{array}$$

Brought over,	1590376
	2783158
	795188
Area of DnC	$=107 \cdot 5094176$
Area of D$or$C 40×12	$=480 \cdot$
	$587 \cdot 5094176$
Mult. by Ov	$=$ 36
	$21150 \cdot 3390336$ (A.)

Again, Ev^2	$= 1408$
$\frac{1}{3}\, ov^2$	$= 133 \cdot 3333333$
	$1274 \cdot 6666666$
Mult. by ov	$=$ 20
	$25493 \cdot 3333333$
Subt. (A.)	$= 21150 \cdot 3390336$
	$4342 \cdot 9942996$
Mult. by $6 \cdot 2832$	23826 inverted.
	260579657976
	8685988599
	3474395440
	130289829
	8685988
Ans.	$27287 \cdot 9017832$ solidity.

(2.) DC$=25$, $sn=\frac{1}{2}(20-15)=2\frac{1}{2}$,

Radius $=\frac{Ds^2+sn^2}{2sn}=\frac{(12\frac{1}{2})^2+(2\frac{1}{2})^2}{5}=32\frac{1}{2}$,

Ov $=32 \cdot 5-10=22 \cdot 5$,

Ev^2 $=(32 \cdot 5)^2-(22 \cdot 5)^2=550$

Subt. $\frac{1}{3}(12 \cdot 5)^2$	$= 52 \cdot 083333$
	$497 \cdot 916666$
Mult. by $12\frac{1}{2}$	$= 12\frac{1}{2}$
Product	$=6223 \cdot 958333$ (A.)

Now since half the chord divided by the radius gives the nat. sine of half the angle at the centre, the same as in the last example, the calculation will be the same, down to the result marked (1.)

∴ ·0397594
Mult. by $R^2=(\frac{65}{2})^2$ = $1056\frac{1}{4}$
Area of D*n*C 41·99586625 product.
Area of D*or*C $25\times7\frac{1}{2}$ = 187·5
Sum, 229·49586625
Mult. by O*v* = $22\frac{1}{2}$
5163·65699062 product.
(A.) 6223·95833333
Diff. 1060·3013427
Mult. by 2382·6 inverted.
1728)6662·0853964 product solid in.
Ans. = 3·85537 solid feet.

PROBLEM XIV.

(1.) $14+6=20\times6^2 = 720\times2{\cdot}4674$,
Gives the solidity $=1776{\cdot}528$,
$(14+6)\times6\times9{\cdot}8696=1184{\cdot}352$, the surface.

(2.) $(31+9)\times9\times9\times2{\cdot}4674=7994{\cdot}376$, solidity.
$(31+9)\times9\times9{\cdot}8696 = 3553{\cdot}056$, surface.

(3.) $(15+5)\times5\times5\times2{\cdot}4674=1233{\cdot}7$, solidity.
$(15+5)\times5\times9{\cdot}8696 = 986{\cdot}96$, surface.

PROBLEM XV.

(1.) Tabular solidity, = ·117851
Multiply by 6^3 = 216
Product, 25·455816, solidity.
Tabular surface, = 1·732051
Multiplied by 6^2 = 36
Product, 62·353836 surface.

(2.) Tabular solidity $1\times4^3 = 64$ solidity.
Tabular surface $=6\times4^2 = 96$ surface.

(3.) Tab. sol. $=7{\cdot}663119\times3^3 = 206{\cdot}904213$, solidity.
Tab. surf. $=20{\cdot}645729\times3^2=$ 185·811561 surface.

(4.) Tab. sol. ·471405 $\times 10^3$ = 471·405, solidity.
Tab. surf. 3·464102 $\times 10^2$ = 346·4102, surface.

(5.) Tab. sol. 2·181695 $\times 8^3$ = 1117·02784, solidity.
Tab. surf. 8·660254 $\times 8^2$ = 554·256256, surface.

LAND SURVEYING.

Problem I.

(1.) 720
609
1044
2)2373
1186·5 Log. of = 3·074268
466·5 „ „ = 2·668852
577·5 „ „ = 2·761552
142·5 „ „ = 2·153815
2)10·658487
2,13424 sq. links = 5·329243
4
·53696
40
21·47840 = 2 acres 21·4784 poles. Ans.

(2.) 925
3
2)2775
Log. 1387·5 = 3·142233
3 Log. 462·5 = 7·995336
2)11·137569
3,70497 sq. links = 5·568784
4
2,81988
40
32·79520 = 3 acres, 2 ro., 32·7952 poles. Ans.

(3.) 546

Mult. by $216 = \frac{1}{2}(432)$

3276

546

1092

1,17936

4

·71744

40

28·69760 = 1 acre, 28·6976 poles. Ans.

Page 101.

(1.) $\frac{1}{2}(256 \times 25) = 3200$

$\frac{1}{2}\{(25+56) \times 239\} = 9679{\cdot}5$

$\frac{1}{2}\{(56+76) \times 105\} = 6930$

$\frac{1}{2}\{(76+48) \times 145\} = 8990$

$\frac{1}{2}(48 \times 228) = 5472$

Offsets on AB 34271·5

$\frac{1}{2}(49 \times 229) = 5610{\cdot}5$

$\frac{1}{2}\{(49+64) \times 125\} = 7062{\cdot}5$

$\frac{1}{2}\ 64+40) \times 182\} = 9464$

$\frac{1}{2}(40 \times 147) = 2940$

Insets on BC =25077

$\frac{1}{2}(55 \times 212) = 5830$

$\frac{1}{2}\{(55+15) \times 212\} = 7420$

$\frac{1}{2}\{(15+72) \times 186\} = 8091$

$\frac{1}{2}\{(72+45) \times 106\} = 6201$

$\frac{1}{2}\{(45+69) \times 108\} = 6156$

$\frac{1}{2}\{(69+37) \times 176\} = 9328$

$\frac{1}{2}(37 \times 252) = 4662$

Offsets on CA =47688

To find the triangle ABC.

AB= 973
BC= 683
CA= 1252
2)2908
1454 Log. of = 3·162564
481 „ „ = 2·682145
771 „ „ = 2·887054
202 „ „ = 2·305351
2)11·037114
△ABC=330033 „ „ = 5·518557
Add 34271·5 offsets on AB.
„ 47688 „ „ CA.
Sum 411992·5
Subtract 25077 insets on BC.
3,86915·5
4
3,47662
40
19·06480; ∴ area = 3 ac., 3 ro., 19·0648 po. **Ans**

(2.) Field Book.

	CA	
0	684	
12	600	
30	500	
65	400	
50	300	
34	200	
20	100	
0	000	
Turn to	R of C	
	BC	
	582	
	469	0
	329	48
	209	0
	000	
Turn to	R of B	

	AB	
0	674	
24	520	
81	360	
46	150	
0	000	
Begin at	A, and	go north.

46 × 150 = 6900
(46 + 81) × 210 = 26670
(81 + 24) × 160 = 16800
24 × 154 = 3696
2)54066
Offsets on AB = 27033 sq. links.
Inset on BC = 260 × 24 = 6240 sq. links.

---

100 × 20 = 2000
(20 + 34) × 100 = 5400
(34 + 50) × 100 = 8400
(50 + 65) × 100 = 11500
(65 + 30) × 100 = 9500
(30 + 12) × 100 = 4200
12 × 84 = 1008
2)42008
Offsets on CA = 21004 sq. links.

Triangle ABC.

AB = 674
BC = 582
CA = 684
2)1940
970 Log. of = 2·986772
296 „ „ = 2·471292
388 „ „ = 2·588832
286 „ „ = 2·456366
2)10·503262
△ABC = 178497 „ „ = 5·251631
27033 offsets on AB.
21004 „ „ CA.
Sum 226534
Subtract 6240 inset on BC.
Carry *2,20294* sq. links, whole area.

Brought 2,20294 sq. links, whole area.
4
·81176
40
32·47040 ∴ area = 2 ac., 0 ro., 32·47 po. Ans

Problem II.

(1). $\frac{(229+388)\times 670}{2} = 617\times 335 = 2{,}06695$ sq. links.
4
·26780
40
10·71200

∴ area = 2 ac., 0 ro., 10·712 poles.

(2.) $\frac{(309+361)\times 984}{2} = 670\times 492 = 3{,}29640$ sq. links.
4
1,18560
40
7·42400

∴ area = 3 ac., 1 ro., 7·424 poles.

(3.) $\frac{(460+340)\times 1020}{2} = 800\times 510 = 4{\cdot}08000$ sq. links.
4
·32000
40
12·80000

∴ area = 4 ac., 0 ro., 12·8 poles.

(4.)
$14\times 170 = 2380$
$(14+38)\times 180 = 9360$
$(38+29)\times 130 = 8710$
$29\times 60 = 1740$
2)22190
Offsets on AB = 11095 sq. links.

$36\times 170 = 6120$
$(36+24)\times 96 = 5760$
$24\times 116 = 2784$
2)14664
Insets on AB = 7332 "

74 × 190 = 14060
(74 + 60) × 205 = 27470
(60 + 40) × 265 = 26500
40 × 156 = 6240
2)74270
Offsets on BC = 37135

86 × 300 = 25800
(86 + 144) × 190 = 43700
144 × 348 = 50112
2)119612
Offsets on CD = 59806

60 × 318 = 19080
(60 + 94) × 262 = 40348
94 × 170 = 15980
2)75408
37704

AB = 922
BC = 1156
CA = 1500
2)3578
1789 Log of = 3.252610
867 ,, ,, = 2·938019
633 ,, ,, = 2·801404
289 ,, ,, = 2·460898
2)11·452931
∴ area △ABC=532678 = 5·7264655

CA = 1500
AD = 1200
DC = 838
2)3538
1769 Log. of = 3·247728
269 ,, ,, = 2·429752
569 ,, ,, = 2·755112
931 ,, ,, = 2·968950
2)11·401542
∴ area △ADC=502078 = 5·700771

Area of △ABC = 532678
" " △ADC = 502078
Offsets on AB = 11095
" " BC = 37135
" " CD = 59806
Sum =1142792
Insets on AB = 7332
" " AD = 37704
Subtract insets = 45036
∴ the whole area = 10,97756 sq. links.
4
3·91024
40
36·40960
∴ the area = 10 ac., 3 ro., 36·4096 perches. Ans.

PROBLEM III.

(1.) 1204 × 524 = 630896
(600 + 454) × 1756 = 1850824
2)2481720
12,40860 sq. links.
4
1·63440
40
25·37600
∴ area = 12 ac., 1 ro., 25·376 p.

(2.) (668 + 120) × 1630 = 1284440
(380 + 440) × 1340 = 1098800
2)2383240
11·91620 sq. links.
4
3·66480
40
26·59200
∴ area = 11 ac., 3 ro., 26·59200 per.

(3.) $(612+534)\times 1360 = 1558560$
$1273\times 600 = 763800$
$1166\times 485 = 565510$
$2)2887870$
14,43935 sq. links.
4
1·75740
40
30·29600

∴ area = 14 ac., 1 ro., 30·296 per.

(4.) $(490+410)\times 1200 = 1080000$
$518\times 1050 = 543900$
$431\times 835 = 359885$
$(431+376)\times 169 = 136383$
$376\times 46 = 17296$
$2)2137464$
10,68732 sq. links.
4
2·74928
40
29·97120

∴ area = 10 ac., 2 ro., 29·9712 per.

PROBLEM IV.

(1.) Offsets on the right.

$(160+240)\times 154 = 61600$
$(240+344)\times 246 = 143664$
$(344+324)\times 226 = 150968$
$(324+274)\times 130 = 77740$
$(274+262)\times 68 = 36448$
$(262+240)\times 128 = 64256$
$(240+154)\times 166 = 65404$

Offsets on the left.

$(200+260)\times 154 = 70840$
$(260+300)\times 126 = 70560$
$(300+316)\times 120 = 73920$
$(316+288)\times 150 = 90600$
$(288+190)\times 274 = 130972$
Carry over, 1036972

△BDC

BD=1930
BC=1340
CD= 950
2)4220

2110	Log. of	= 3·324282
180	„ „	= 2·255273
770	„ „	= 2·886491
1160	„ „	= 3·064458
		2)11·530504
△BDC = 582441		= 5·765252

$$\triangle ABD = 891250 = \frac{1550\times1150}{2}$$

1473691
Subt. 186750, sum of the insets.
12·86941 sq. links.
4
3·47764
40
19·10560

∴ area = 12 ac., 3 ro., 19·1 poles.

Problem VIII.

△BAC		
	Log. 252	= 2·401401
	Log. 445	= 2·648360
	Log. sin. 94° 30′	= 9·998453
∴ area=111741		= 5·048214

△CAD		
	Log. 445	= 2·648360
	Log. 162·5	= 2·210853
	Log. sin. 44° 53′	= 9·848599
∴ area =51028·5		= 4·707812

△DAE		
	Log. 162·5	= 2·210853
	Log. 201	= 2·303196
	Log. sin. 70° 28′	= 9·974257
∴ area =30783·6		= 4·488306

$$\triangle EAF \begin{cases} \text{Log. } 201 & = 2{\cdot}303196 \\ \text{Log. } 324 & = 2{\cdot}510545 \\ \text{Log. sin. } 92^\circ\ 2' & = \underline{9{\cdot}999726} \end{cases}$$

$\therefore$ area = 65082·8 = 4·813467

$$\triangle FAB \begin{cases} \text{Log. } 324 & = 2{\cdot}510545 \\ \text{Log. } 504 & = 2{\cdot}702431 \\ \text{Log. sin. } 57^\circ\ 47' & = \underline{9{\cdot}927390} \end{cases}$$

$\therefore$ area = 138155 = 5·140366

$$\triangle FBG \begin{cases} \text{Log. } 572 & = 2{\cdot}757396 \\ \text{Log. } 202 & = 2{\cdot}305351 \\ \text{Log. sin. } 79^\circ\ 5' & = \underline{9{\cdot}992069} \end{cases}$$

$\therefore$ area = 113453 = 5·054816

$$\triangle GBH \begin{cases} \text{Log. } 202 & = 2{\cdot}305351 \\ \text{Log. } 417 & = 2{\cdot}620136 \\ \text{Log. sin. } 11^\circ\ 50' & = \underline{9{\cdot}311893} \end{cases}$$

$\therefore$ area = 17273·5 = 4·237380

$$\triangle HBL \begin{cases} \text{Log. } 417 & = 2{\cdot}620136 \\ \text{Log. } 70 & = 1{\cdot}845098 \\ \text{Log. sin. } 68^\circ\ 58' & = \underline{9{\cdot}970055} \end{cases}$$

$\therefore$ area = 27245·1 = 4·435289

$$\begin{array}{lcr} \therefore \triangle BAC & = & 111741 \\ \triangle CAD & = & 51028{\cdot}5 \\ \triangle DAE & = & 30782{\cdot}6 \\ \triangle EAF & = & 65082{\cdot}8 \\ \triangle FAB & = & 138155{\cdot} \\ \triangle FBG & = & 113453 \\ \triangle GBH & = & 17273{\cdot}5 \\ \triangle HBL & = & \underline{27245{\cdot}1} \\ \therefore \text{the whole area} & = & 5{\cdot}547615 \text{ acres.} \\ & & \underline{4} \\ & & 2{\cdot}190460 \\ & & \underline{40} \end{array}$$

$\therefore$ Or 5 acres, 2 ro., 7·618400 po. Ans.

PROBLEM IX.

(1.) R = 10·000000
Cos. 21° 14′ = 9·969469
Surf. line 1560 Log. = 3·193125
∴ Base line =1454·1 Log. = 3·162594

(2.) R = 10·000000
Cos. 9° 16′ = 9·994295
Surf. line 500 Log. = 2·698970
Base line =493·47 Log. = 2·693265

R = 10·000000
Cos. 18° 33′ = 9·976830
Surf. line 900 Log. = 2·954243
Base line =853·24 Log. = 2·931073

R = 10·000000
Cos. 23° 10′ = 9·963488
Surf. line 700 Log. = 2·845098
Base line =643·55 Log. = 2·808586
∴ the whole base =493·47+853·24+643·55=1990·26. Ans.

(3.) R = 10·000000
Cos. 21° = 9·970152
Surf. line 900 = 2·954243
Base line =840·22 Log. = 2·924395

R = 10·000000
Cos. 17° 12′ = 9·980130
Surf. line 900 Log. = 2·954243
Base line =859·75 Log. = 2·934373
∴ the whole base =840·22+859·75=1699·97. Ans.

PROBLEM XI.

(1.) Log. 31576 = 4·499347
Log. 1800 = 3·255273
7·754620
Subtract const. Log. = 7·319890
Correction =2·721 = ·434730
∴ 31576—2·721 =31573·279. Ans.

(2.) 5 miles =26400 feet.
Log. 26400 = 4·421604
Log. 821 = 2·914343
7·335947
Subtract const Log. = 7·319890
Correction 1·038 feet = ·016057
5 miles —1·038 feet =4 miles, 1759 yds., 1·962 feet.

PROBLEM XII. CASE I.

(1.) One acre = 100000 square links, which being divided by 400 links, the breadth, gives 250 links, the length required.

(2.) 5 acres, 3 roods, and 10 poles =581250 sq. links, the square root of which is 762·4 nearly, the side of the square. Again, 1 acre, 3 roods, 5 poles =178125 square links, which being divided by 762·4, gives 233·6, the length required.

CASE II.

(1.) L.3, 12s. = 72s.
5 15 = 115
7 5 = 145
6 8 = 128
460 : 72 : : 1380 : 216.
460 : 115 : : 1380 : 345.
460 : 145 : : 1380 : 435.
and 460 : 128 : : 1380 : 384.

(2.) $2+5+7+9+11=34$
34 : 2 : : 1530 : 90
34 : 5 : : 1530 : 225
34 : 7 : : 1530 : 315
34 : 9 : : 1530 : 405
34 : 11 : : 1530 : 495

CASE III.

(1.) $500\times2100=1050000$ sq. links $=10\frac{1}{2}$ acres.
$\therefore 10\frac{1}{2} : 9\frac{1}{2} : : (1000)^2 : 904762$, the square root of which is 951·2 *links nearly.*

(2.) Since the part cut off is 1½ acre, the remaining part will be 9 acres; therefore

10½ : 9 :: $(1000)^2$: 857142·85, &c., the square of the distance of the point from the hypotenuse; hence extracting the square root, we have 925·8, the distance from the other extremity of the base, which being subtracted from 1000, gives 74·2, the distance sought.

(3.) Since the part cut off is 2½ acres, the part left is 8 acres; therefore

10½ : 8 :: $(1000)^2$: 761904·76, the square root of which is 872·9.

Again, 10½ : 8 :: $(2100)^2$: 3360000, the square root of which is 1833.

Miscellaneous Exercises.

(1.) 174 × 6½ = 1131 square yards, and
4840 : 1131 :: L.10, 10s. : L2, 9s. 0¾d.

(2.) 454 × 62 = 28148 sq. links.

$$\begin{array}{r} 4 \\ \hline 1·12592 \\ 40 \\ \hline 5·03680 \end{array}$$

Also 100000 : 28148 :: L.24 : L.6, 15s. 1¼d.

(3.) 272 × 30 = 8160,
And 100 : 8160 :: 3s. 6d. : L.14, 5s. 7⅓d.

(4.) $\frac{196\times60\times2}{3}=7840$ yards.
And 4840 : 7840 :: L.21, 15s. : L.35, 4s. 7½d.

(5.) Half an acre = 2420 square yards, which being divided by 5½, gives for quotient 440 yards, the length required.

$$\begin{array}{lcr} \text{(6.) Log. 654} & = & 2·815578 \\ & & 2 \\ \hline & & 5·631156 \\ \text{Subtract Log. 2} & = & 301030 \\ \hline \text{Carry over} & & 5·330126 \end{array}$$

Brought over 5·330126
Log. cosec. 122° 35′ = 10·074374
Log. sin. 58° 20′ = 9·929989
Log. sin. 64° 15′ = 9·954579
1,94566 = 5·289068
4
3,78264
40
31,30560 = 1 acre, 3 ro., 31·3056 poles. Ans.

(7.) $2500\times 700=1750000=17\frac{1}{2}$ acres.
And L.43, 15s.$\div 17\frac{1}{2}=$L.2, 10s. Ans.

(8.) By Geometry, Proposition 68, cor. 1., we have 880 : 660 : : 660 : 495, the other segment of the hypotenuse; therefore

$880\times 330=290400=$ 2 ac., 3 ro., 24·64 poles.
And $495\times 330=163350=$ 1 ac., 2 ro., 21·36 poles.

(9.) Let $x=$ the breadth of the walk in feet; then
$200x+2(80-2x)x=\frac{1}{2}\times 100\times 80$, by the question.
$4x^2-360x=-4000$, by changing the signs.
$x^2-90x=-1000$, dividing by 4.
$x^2-90x+(45)^2=1025$, completing the square.
$x-45=\pm 32·0156$, extracting the root.
$\therefore x=12·9844$. Ans.

(10.) 25—9=16, the remaining part of AC.
And 25 : 30 : : 16 : 19·2, the remaining part of BC.
Also 25 : 40 : : 16 : 25·6, side parallel to AB.

(11.) Let the length of the required side be x; then since the area of the triangle cut off is to be half of the original triangle, and the angle A is common, we have the following equation:

$400\times x \sin. A=\frac{1}{2}\times 650\times 560 \sin. A$;
Hence $400x=182000$.
$\therefore x=455$.

(12.) To find △ABC.
1194
926
Carry over, *536*

Brought over, 536
2|2656
1328 Log. of = 3·123198
134 „ „ = 2·127105
402 „ „ = 2·604226
792 „ „ = 2·898725
2|10·753254
Area, 238027 „ „ = 5·376627

To find △ACE.

1223
1194
587
2|3004
1502 Log. of = 3·176670
279 „ „ = 2·445604
308 „ „ = 2·488551
915 „ „ = 2·961421
2|11·072246
Area 343655 „ „ = 5·536123

To find △ECD.

1223
835
628
2|2686
1343 Log. of = 3·128076
120 „ „ = 2·079181
508 „ „ = 2·705864
715 „ „ = 2·854306
2|10·767427
Area, 241943 „ „ = 5·383713
823625, whole area; and
100000 : 823625 : : L.2, 5s. : L.18, 10s. 7½d. Ans.

KEY.—SPECIFIC GRAVITIES.

Problem I.

(1.) Spec. gr. mahogany = 1063

$35\frac{1}{8}$

$16\overline{)37337\frac{7}{8}}$ oz.

$28\overline{)2333 : 9\frac{7}{8}}$

$4\overline{)83 : 9 : 9\frac{7}{8}}$

$20\overline{)20 : 3 : 9 : 9\frac{7}{8}}$

Ton $\overline{1 : 0 : 3 : 9 : 9\frac{7}{8}}$ **Ans.**

(2.) Spec. gr. French box = 912

Multiplied by $3\frac{1}{2}$

Ans. 1 cwt., 3 qrs., 3 lb., 8 oz. = 3193 oz.

(3.) Spec. gr. cast iron = 7207

Mult. by $\frac{450}{1728}=\frac{25}{96}$ = $\frac{25}{96}$

Ans. 1 cwt., 5 lb. 5 oz. = 1877 nearly.

(4.) Spec. gr. of beech = 852

Mult. by $20\times(1\frac{1}{2})^2$ = 45 feet in the log.

4260

3408

Ans. 1 ton, 1 cwt. 1 qr. 16 lb. 4 oz. = 38340 oz.

(5.) $(3\frac{1}{4})^2\times{\cdot}7854\times15\frac{5}{12}\times{\cdot}2654=$ wt. in ounces.

2 Log. $3\frac{1}{4}$ = 1·023766

Log. ·7854 = $\bar{1}$·895091

Log. $15\frac{5}{12}$ = 1·187991

Log. 2654 = 3·423901

339429 = 5·530749

And 339429 oz. = 9 tons, 9 cwt. 1 qr. 18 lb. 5 oz. **Ans.**

(6.) $\frac{1}{144}(30+27\frac{1}{2})\times2\frac{1}{2}\times{\cdot}7854\times5=3{\cdot}92018$ ft. solid.

Mult. by spec. gr. = 7207

2744126

784036

2744126

28252·73726

And 28253 = 15 cwt., 3 qrs., 1 lb., 13 oz. **Ans.**

PROBLEM II.

(1.) 2 tons 7 cwt. 13 lb. 12 oz. = 84444 oz.
And 84444÷2724= 31 feet. Ans.

2.) 15 cwt. 2 qrs. 6 lb. 2 oz. = 27874 oz.
And 27874÷1991=14 feet. Ans.

(3.) 7 cwt. 1 qr. 24 lb. 4 oz. =13380.
And 13380÷2676= 5feet. Ans.

(4.) 1 ton =35840 oz.
And 35840÷1000=35·84 ft. Ans.

(5.) 5 lb. 7 oz.
16
87
1000 = ·087 feet.
1728
12096
13824
150·336 cubic inches. Ans.

PROBLEM III.

(1.) 7 cwt. 3 qrs. 26 lb. 15 oz. =14319 oz.
And 14319÷4·5=3182= the specific gravity green limestone.

(2.) 1 cwt. 8 lb. 12 oz. =1932 oz.
And 1932÷3=644= the specific gravity Spanish cypress.

(3.) 1 lb. 10 oz. $4\frac{1}{4}$ dr. =$26\frac{17}{64}$ oz.
And $26\frac{17}{64}\times 1728\div 4\frac{1}{3}$=10474 nearly. Ans.

(4.) 35 lb. 3 oz. 6 dr. = $563\frac{3}{8}$ oz.
Mult. by 1728
5×5×5= 125)973512
7788·096, which is specific gravity of hammered copper, or bar iron.

(5.) $6\times\frac{1}{3}\times\frac{1}{6}=\frac{1}{3}$ foot solidity.

cwt.	qr.	lb.	oz.
1	1	22	4

$$\begin{array}{r} 1 \\ 4 \\ \hline 5 \\ 28 \\ \hline 162 \\ 16 \\ \hline \frac{1}{3})2596 \\ \hline 7788 \end{array}$$

7788 spec. gr. of hammered copper or bar iron.

Problem IV. Case I.

(1.) $50-42=8$.

$\therefore 8:50::1000:$ spec. gr.

$$\begin{array}{r} 50 \\ \hline 8)50000 \\ \hline 6250 \end{array}$$

6250 spec. gr. Ans.

(2.) $15-14=1$.

$\therefore 1:15::1000:15000$. Ans.

(3.) $26-16=10$.

$\therefore 10:26::1000:2600$. Ans.

(4.) $4-3=1$.

$\therefore 1:4::1000:4000$, sp. gr. of girasol.

(5.) $15-14\frac{1}{16}=\frac{15}{16}$.

$\therefore \frac{15}{16}:15::1000:16000$.

Case II.

(1.) $20+12=32$, the weight of the compound mass.
$12-10\frac{1}{2}=1\frac{1}{2}$, weight lost by the metal.
$32-4\frac{1}{2}=27\frac{1}{2}$, weight lost by the mass.

$27\frac{1}{2}-1\frac{1}{2}=26:20::1000$

$$\begin{array}{r} 20 \\ \hline 26)20000 \\ \hline 769\frac{5}{13} \end{array}$$

$769\frac{5}{13}$. Ans.

(2.) 14+11=25, the weight of the compound mass.
11—10= 1, weight lost by the metal.
25— 2=23, weight lost by the mass.
23—1 =22 : 14 :: 1000
11 : 7 7
11)7000
636$\frac{4}{11}$. Ans.

(3.) 20+16=36, the weight of the compound mass.
16—14= 2, weight lost by the metal.
36— 1=35, weight lost by the compound mass.
35— 2=33 : 20 :: 1000
20
33)20000
606$\frac{2}{33}$. Ans.

(4.) 14+ 3 =17, weight of the compound mass.
14—12½= 1½, weight lost by the metal.
17— 3 =14, weight lost by the compound mass.
14— 1½=12½ : 3 :: 1000
2 2 6
25 6 25)6000
240. Ans.

CASE III.

lb. oz. dr.
(1.) 7 9 11
6 11 0
lb. oz. dr.
7 9 11 : 14 11 :: 7788
16 16 235
121 235 1947)1830180
16 940. Ans.
1947

(2.) lb. oz. dr.
2 9 11
1 6 5
lb. oz. dr.
2 9 11 : 1 3 6 :: 2668
16 16 310
41 19 26680
16 16 8004
667 310 667)827080
1240. Ans.

$$\begin{array}{lrrrcrcr}
 & \text{lb.} & \text{oz.} & \text{dr.} & & \text{oz.} & & \\
(3.) & 2 & 8 & 9 & : & 5 & :: & 7788 \\
 & \underline{16} & & & & \underline{16} & & \underline{80} \\
 & 40 & & & & 80 & & 649)\overline{623040} \\
 & \underline{16} & & & & & & \overline{960.}\ \text{Ans.} \\
 & 649 & & & & & &
\end{array}$$

PROBLEM V.

(1.) Sp. gr. of gold = 19258
Sp. gr. of silver = 10474

$$\frac{85\times19258\times(15800-10474)}{(19258-10474)\times15800}=62{\cdot}8177, \text{ the gold.}$$

$$\frac{85\times10474\times(19258-15800)}{(19258-10474)\times15800}=22{\cdot}1823, \text{ the silver.}$$

(2.) Sp. gr. of silver = 10474
Sp. gr. of copper = 7788

$$\frac{100\times10474\times(8530-7788)}{(10474-7788)\times8530}=33{\cdot}92, \text{ the silver.}$$

$$\frac{100\times7788\times(10474-8530)}{(10474-7788)\times8530}=66{\cdot}08, \text{ the copper.}$$

(3.) Sp. gr. of platina = 19500
Sp. gr. of silver = 10474

$$\frac{80\times19500\times(16420-10474)}{(19500-10474)\times16420}=62{\cdot}5865+ \text{ the platina.}$$

$$\frac{80\times10474\times(19500-16420)}{(19500-10474)\times16420}=17{\cdot}4135- \text{ the silver.}$$

KEY—GAUGING.

MISCELLANEOUS EXERCISES, Page 159.

(1.) From Table, page 131, the divisor for a circle is 11, which being multiplied by 3, (Art. 4), gives 33, the divisor. Now $1\div33=\cdot030303$, the multiplier; and the *square root of 33=5·7445*, the gauge point.

(2.) From Table, page 131, the number of square inches in a bushel is 2218·192, which being (Art. 4) divided by 1·720477, the multiplier for a pentagon, (page 67), gives 1289·2889, the divisor; and 1÷1289·2889= ·0007756, the multiplier; and the square root 1289·2889= 35·9067, the gauge point.

(3.) The circular measure of a gallon, by Table, page 131, is 353·0363, which being multiplied by 6, gives 2118·2178, the square root of which is 46·024, the gauge point required.

(4.) To 1 on A set ·27 on B, then opposite to 74 on A will be 1·998 on B.

(5.) The area in bushels 7·44, multiplied by 18·3, the depth, gives the content 136·15 bushels, which (Art. 33) being multiplied by 81·5, and the product divided by 100, gives 110·96, the net malt.

(6.) $\frac{766\times164\times3{\cdot}1}{2218{\cdot}192}=175{\cdot}5$ bush. Ans.

(7.) $45\times15\times51{\cdot}4=34695$, which being divided by 24·04, (Table, page 131), gives 1443·2 lb. Ans.

(8.)

$$\begin{array}{lrrr}
\text{Cross,} & 24{\cdot}2 & 25{\cdot}3 & 27{\cdot}4 \\
\text{Diameters,} & \underline{23{\cdot}9} & \underline{25{\cdot}5} & \underline{28{\cdot}1} \\
 & 2\overline{)48{\cdot}1} & 2\overline{)50{\cdot}8} & 2\overline{)55{\cdot}5} \\
\text{Mean dia.} & 24{\cdot}0 & 25{\cdot}4 & 27{\cdot}7
\end{array}$$

$$\therefore\frac{(24^2+25{\cdot}4^2+27{\cdot}7^2)\times7}{14{\cdot}3876}=\frac{(576+645{\cdot}16+767{\cdot}29)\times7}{14{\cdot}3876}$$

$$\frac{1988{\cdot}45\times7}{14{\cdot}3876}=\frac{13919{\cdot}15}{143876)13919{\cdot}15(967{\cdot}44 \text{ lb.}}\quad\text{Ans.}$$

$$\begin{array}{r}
\underline{1294884} \\
970310 \\
\underline{863256} \\
1070540 \\
\underline{1007132} \\
634080 \\
\underline{575504} \\
585760 \\
\underline{575504}
\end{array}$$

(9.) By Rule, page 153,

$\frac{250}{2}=125=$ the floor net.

$158{\cdot}8\times81{\cdot}5\div100=129{\cdot}42$, the couch net, which being the greater, is that on which the duty is charged.

(10.) By Rule, page 156,

$36{\cdot}5-28{\cdot}2=8{\cdot}3$, and $8{\cdot}3\times{\cdot}7=5{\cdot}8$.

$\therefore\ 28{\cdot}2+5{\cdot}8=34{\cdot}0=$ the mean diameter.

Hence $34^2\times{\cdot}002832\times43=140{\cdot}7$ gall.

Or by the Sliding Rule; having found the mean diameter as before, set the length 43 on B to 18·7892 on D, and opposite to the mean diameter 34 on D is 141 nearly, the content on B.

(11.) By the sliding rule, (Art. 42), set the bung diameter 33·5 on C, to 100 on the line marked SL, and opposite to 24·1, the wet inches on C, is 78·5, the segment, or mean area on the line marked SL. Then having set the whole content, 108 on B, to 1 on A, opposite to 78·5 on A is found 84·5, the ullage quantity on B.

The same by the pen.

$24{\cdot}1\div33{\cdot}5={\cdot}719$, and ${\cdot}719-{\cdot}500={\cdot}219$.

Again, ${\cdot}719+\frac{{\cdot}219}{4}={\cdot}719+{\cdot}055={\cdot}774$.

And ${\cdot}774\times108=83{\cdot}592$, or 83·6 nearly.

(12.) Set the divisor, 57 on B, to 1 on A; and opposite to 228 on B will be found the quotient, 4 on A.

(13.) $\frac{70-50{\cdot}4}{4}=\frac{19{\cdot}6}{4}=4{\cdot}9$, variation of the diameters for 10 inches, and $4{\cdot}9\div2=2{\cdot}45$, variation for 5 inches of depth. Therefore the diameter at the middle of the first 10 inches is $50{\cdot}4+2{\cdot}45=52{\cdot}85$. To this add 4·9, and sum is 57·75 inches, the diameter at the middle of the second 10 inches of depth; and in the same manner it is found, that the diameters at the middle of the third and fourth 10 inches are 62·65 and 67·55.

To find the mean area in gallons of each 10 inches. Square the diameter at the middle of the section, and divide the square by 353·0363, the tabular divisor for a *circle; thus,*

$(52·85)^2 \div 353·0363 = 7·9117$ gal., 1st mean area.
$(57·75)^2 \div 353·0363 = 9·4468$ gal., 2d mean area.
$(62·65)^2 \div 353·0363 = 11·1122$ gal., 3d mean area.
$(67·55)^2 \div 353·0363 = 12·9250$ gal., 4th mean area.

These calculations being made, the areas are then tabulated, by considering the mean area as constant for 10 inches; the first area is the content when the liquor is one inch deep, this added to itself, gives the content for 2 inches in depth, and so on up to 10 inches in depth; we then commence to add the next area, till we have got to a depth of 20 inches, when we commence with the third, and so on.

Wet inches.	Areas in galls.	Wet inches.	Areas in galls.
1	7·9117	21	184·6972
2	15·8234	22	195·8094
3	23·7351	23	206·9216
4	31·6468	24	218·0338
5	39·5585	25	229·1460
6	47·4702	26	240·2582
7	55·3819	27	251·3704
8	63·2936	28	262·4826
9	71·2053	29	273·5948
10	79·1170	30	284·7070
2dar ea,	9·4468	4th area,	12·9250
11	88·5638	31	297·6320
12	98·0106	32	310·5570
13	107·4574	33	323·4820
14	116·9042	34	336·4070
15	126·3510	35	349·3320
16	135·7978	36	362·2570
17	145·2446	37	375·1820
18	154·6914	38	388·1070
19	164·1382	39	401·0320
20	173·5850	40	413·9570
3d area,	11·1122		Full.

And 414 gallons = 6 qrs. 3 bush. 6 gal.

(14.) $72 \times 50 \times ·002832 = 10·1952$ galls.

(15.) $\frac{24^2+36^2+24\times36}{3} \times 53 \times ·000354 = 17·11$ bush.

(16.) Here $(3·8+3·6+4·4+4·4+3·9+4·0+3·5+4·2+4·3+3·9) \div 10 = 4$, the mean depth.

And $(212 \times 148 \times 4) \div 277·274 = 452·63$ galls.

(17.) Cross	75·0	74·8	69·6	65·5	61·6	57·1
Diameters,	74·6	74·4	69·0	65·5	61·2	57·2
2	149·6	149·2	138·6	131·0	122·8	114·3
Mean. diam.	74·8	74·6	69·3	65·5	61·4	57·1

$(74\cdot8)^2\div353\cdot0363=15\cdot8483$ gal., 1st area.
$(74\cdot6)^2\div353\cdot0363=15\cdot7637$ gal., 2d area.
$(69\cdot3)^2\div353\cdot0363=13\cdot6034$ gal., 3d area.
$(65\cdot5)^2\div353\cdot0363=12\cdot1524$ gal., 4th area.
$(61\cdot4)^2\div353\cdot0363=10\cdot6787$ gal., 5th area.
$(57\cdot1)^2\div353\cdot0363=9\cdot2353$ gal., 6th area.

The whole may be tabulated as follows:—

Depths.	Inches from top.	Cross diameters.		Mean diam.	Areas in gallons.	Contents in gallons.
12	6	75·0	74·6	74·8	15·8483	190·1796
10	17	74·8	74·4	74·6	15·7637	157·6370
8	26	69·6	69·0	69·3	13·6034	108·8272
6	33	65·5	65·5	65·5	12·1524	72·9144
6	39	61·6	61·2	61·4	10·6787	64·0722
6	45	57·1	57·2	57·1	9·2353	55·4118
48 depth.				To cover the crown,		22·

9)671·0422
4)74–5·04 gal.
Whole content, 18 B. 2 F. 5·04 g.

(18.) Wet inches $19\cdot8\div46=\cdot430$.
$\cdot500-\cdot430=\cdot070\div10=\cdot007$.
And $\cdot430-\cdot007=\cdot423$.
Then $\cdot423\times207=87\cdot561=87\frac{1}{2}$ nearly.

(19.) $(29\cdot9)^2=894\cdot01\div11 = 81\cdot2736$ lb. Area.
Multiply by the depth $= 18\frac{1}{2}$
406368
6501888
812736
1503·5616 lb. Content.

(20.) $(32\cdot5)^2=1056\cdot25$.
And $1056\cdot25\div277\cdot274=3\cdot8094$, 1st area.
$(38\cdot5)^2=1482\cdot25$.
And $1482\cdot25\div277\cdot274=5\cdot3458$, 2d area.
$(42\cdot5)^2=1806\cdot25$.
And $1806\cdot25\div277\cdot274=6\cdot5143$, 3d area.

(21.) $19·6-17·4=2·2$, and $2·2\times·68=1·5$.
Hence $17·4+1·5=18·9$, the mean diameter.
$(18·9)^2\times·002832\times23·7=23·97$, &c. $=24$ gal. nearly.

By the Sliding Rule.

Set 23·7 on B to 18·7892 on D, and opposite to 18·9, the mean diameter on D, is 24 gallons on B.

(22.) $863\times285\times1·7\div2218·192=188·5$ bush.

(23.) $(36·5)^2\times71·4\div35·6557=2667·81$ lb.

(24.) First, to find the globular part by Mensuration of Solids, Problem XI.

Cross	27·2	55·2
Diameters,	26·8	54·8
	4)54	110
Mean radii,	13·5	27·5

$3\{(13·5)^2+(27·5)^2\}+8^2=2879·5$, and
$2879·5\times8\times·5236=12061·6496$, content in inches.
Again, $12061·6496\div277·274=43·5008$ gal., the content of the spherical part.

Next, to find the body of the still, tabulate as follows:

Depths.	Inches from top.	Cross diameters.		Mean diam.	Area in gallons.	Content in gallons.
9	4·5	59·8	60·2	60·0	10·1972	91·7748
9	13·9	63·8	64·4	64·1	11·6385	104·7465
9	22·5	64·0	64·6	64·3	11·7112	105·4008
11	32·5	62·0	62·4	62·2	10·9587	120·5457
38	Depth.		Content of globular part,			43·5008
			To cover the crown,			35·
			Whole content in gallons,			500·9686

KEY—GENERAL EXERCISES.

(1.) $560\times426 = 238560$
$280\times213 = 59640$
Difference $1,78920=1$ ac. 3 ro. 6·272 poles.
4
3,15680
40
6·27200

$$\begin{array}{llr} (2.)\ 12^2 & = & 144 \\ \ 15^2 & = & 225 \\ \ 16^2 & = & \underline{256} \\ \ \text{Sum} & & 625, \end{array}$$

the square root of which being 25, is the length of the side required.

(3.) $(99)^2=9801$, which divided by 9, gives 1089, the square root of which is 33, the Ans.

Otherwise, let $x=$ the side required; then

$$1:\tfrac{1}{9}=99^2:x^2.$$
$$\therefore 1:\tfrac{1}{3}=99\ :x.\quad \text{(Alg. 110.)}$$
$$\text{Hence } x=33.\quad \text{Ans.}$$

(4.) The sides being 33 and 11 yards respectively, we have, by Mensuration of Surfaces, Prob. VI.,

$$\begin{array}{lcr} \text{Tabular area} & = & \cdot433013 \\ \text{Mult. by } (33^2+11^2) & = & \underline{1210} \\ & & 4330130 \\ & & \underline{5196156} \\ & & 523\cdot945730 \ \text{Ans.} \end{array}$$

(5.) Since the length is 5 feet 3 in. and the width 3 feet 2 in., the perimeter is 16 feet 10 in. $=202$ inches, which being multiplied by 30 in., the depth, gives the area of the sides and ends $=6060$ square inches; to which add the area of the top and bottom $=63\times38\times2=4788$ square inches; therefore the whole surface $=10848$, and $10848\div16=678$ inches, the length of the paper $=18\frac{5}{6}$ yards. Ans.

(6.) $\sqrt{9^2+12^2}=\sqrt{225}=15$ feet, the slant.

$\therefore 3\cdot1416\times9\times15=424\cdot116$ sq. feet $=47\cdot124$ sq. yards.

(7.) Let $3x$ be one of the sides, then $4x$ will be the other, since they are to one another as 3 to 4; and $5x$ will be the hypotenuse; and hence $12x$ will be the perimeter, and $6x^2$ the area.

Then $6x^2=12x\times2\frac{1}{2}$, by the question.

$\therefore 6x=30.$

And $x=5.$

Hence $3x=15$, $4x=20$, and $5x=25$, the Ans.

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ft. in. ft. in. sq. feet.
(8.) $30:3\times15:6=468\frac{126}{144}$.
lb. lb. cwt. qrs. lb. oz.
$\therefore 9\times468\frac{126}{144}=4219\frac{14}{16}=37:2:19:14$. Ans.

(9.) Let x = the width of the frame.
$\therefore 30+2x=$ the length of the frame.
$24+2x=$ the breadth of the frame.
$(30+2x)(24+2x)=720+108x+4x^2=$ the surface of the glass and frame.
$30\times24=720$, surface of the glass.
$\therefore$ their dif. $=108x+4x^2=$ surf. of the frame.
Hence $4x^2+108x=\frac{3}{4}$ of $720=540$, by the question.
$x^2+27x=135$, div. by 4.
$4x^2+108x+27^2=1269$, completing the square.
$2x+27=35{\cdot}623026$, extracting the root.
$\therefore x=4{\cdot}311513$, by transp. and divid. by 2.

(10.) $1200\times240\times12=345600$, solidity of the wall in inches.
$9\times4\frac{1}{2}\times3=121\frac{1}{2}$, solidity of a brick in inches.
$\therefore \frac{345600}{121\frac{1}{2}}=28444\frac{4}{9}$, the number of bricks.

(11.) Since 36 feet =12 yards, and the perimeter of the square is 12 yards, its side =3 yards; also since the perimeter of the triangle =12 yards, its side =4 yards; hence
$\cdot0795775\times12^2=11{\cdot}45916$, the area of the circle.
$3^2=9$, the area of the square.
$\cdot433013\times4^2=6{\cdot}929208$, the area of the triangle.

(12.) $\cdot7854\times(100)^2=7854$, area of the circle.
$(100)^2=10000$, area of the square.
$\cdot43013\times(100)^2=4330{\cdot}13$, area of the triangle.

(13.) Let $x=$ the quarters of wheat,
And $y=$ the quarters of barley; then
$55x=33y$; hence $5x-3y=0$.
Also $55x+33y=$ the rent in shillings,
And $65x+41y=55x+33y+140$, by the question.
$10x+8y=140$, by transposition.
$10x-6y=0$, by the first equation.

$\therefore 14y=140$, and hence
$y=10$, the quarters of barley.
$55x=330$, by the first equation.
$\therefore x=6$, the quarters of wheat.

(14.) Let $x=$ the number of revolutions of the hind wheel.

Then $x+n=$ the number of revolutions of the fore wheel.

Also $bx=$ the distance in feet, and
$a(x+n)$ also $=$ the distance in feet; hence
$bx=ax+an$,
$bx-ax=an$;

$$\therefore x=\frac{an}{b-a};$$

Wherefore $bx=\frac{abn}{b-a}$, the distance sought.

$\therefore$ the distance$=\frac{10\frac{1}{2}\times12\times1000}{12-10\frac{1}{2}}=84000$ feet $=15$ miles 7 fur. 60 yds.

(15.) Let $S=$ the sum of the 4 numbers.

$$\therefore \frac{252}{S}=\text{ the 1st.}$$

$$\frac{504}{S}=\text{ the 2d.}$$

$$\frac{396}{S}=\text{ the 3d.}$$

$$\frac{144}{S}=\text{ the 4th.}$$

$$\therefore \frac{252+504+396+144}{S}=S\text{, the sum.}$$

Hence $S^2=1296$.
$\therefore S=36$, their sum, and hence

The first $=\frac{252}{36}=7$.

The second $=\frac{504}{36}=14$.

The third $=\frac{396}{36}=11$.

The fourth $=\frac{144}{36}=4$.

(16.) Let $x=$ the number of half-crowns.
$y=$ the number of shillings.
$z=$ the number of sixpences.

Then	1	$y+z=409$	By the question.
	2	$x+y=1254$	
	3	$x+z-42=1103$	
(1)+(2)+(3)	4	$2x+2y+2z-42=2766.$	
(4) transp.	5	$2x+2y+2z=2808.$	
(5)÷2	6	$x+y+z=1404.$	
(6)—(1)	7	$\therefore x=995$, the half-crowns.	
(6)—(3)	8	$y=259$, the shillings.	
(6)—(2)	9	And $z=150$, the sixpences.	

Again,		L.	s.	d.
995 half-crowns	=	L.124	7	6
259 shillings	=	12	19	0
150 sixpences	=	3	15	0
Value of the whole	=	L.141	1	6

(17.) Let $x=$ the one fraction,
and $y=$ the other; then

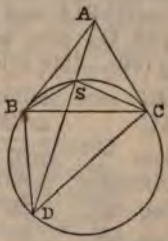
	1	$x+y=\frac{1}{7}$	By the question.
	2	$xy=\frac{4}{847}$	
(1) squared	3	$x^2+2xy+y^2=\frac{1}{49}.$	
(2)×4	4	$4xy=\frac{16}{847}.$	
(3)—(4)	5	$x^2-2xy+y^2=\frac{9}{5929}.$	
sq. root of (5)	6	$\therefore x-y=\frac{3}{77}.$	
(1)+(6)	7	$2x=\frac{14}{77}.$	
(7)÷2	8	$x=\frac{7}{77}=\frac{1}{11}$ the one fraction.	
(1)—(6)	9	$2y=\frac{8}{77}.$	
(9)÷2	10	$y=\frac{4}{77}=$ the other fraction.	

(18.) $100-\sqrt{(100)^2-(10)^2}=100-\sqrt{9900}=$
$100-99{\cdot}49874 \quad = {\cdot}50126$ feet.
$\quad\quad 12$
$6{\cdot}01512$ inches.

(19.) Since the part broken off struck the ground at the distance of 30 feet from the bottom, it is evident that it formed the hypotenuse of a right-angled triangle, of which the base was 30 feet, and the perpendicular was the other part of the pole; therefore the part left standing $=\sqrt{(63)^2-(30)^2}=\sqrt{(63+30)(63-30)}=\sqrt{93\times 33}=$ *55·3985559*; to which add 63, the part broken off, and the *sum is* 118·3985559. Ans.

(20) Let the four numbers be represented by $x-6$, $x-2$, $x+2$, and $x+6$, whose differences are 4; then by the question, $(x-6)(x+6)\times(x-2)(x+2)=(x^2-36)(x^2-4)=x^4-40x^2+144=176985$.

$x^4-40x^2 \quad =176841$, by transposition.
$x^4-40x^2+400=177241$, completing the square.
$x^2-20=\pm 421$, extracting the root.
$x^2=20+421=441$.
$\therefore x=21$.
Hence $x-6=15$, the first.
$x-2=19$, the second.
$x+2=23$, the third.
$x+6=25$, the fourth.



(21.) Let B represent the Lizard, A Plymouth, C the Start, and S Eddystone Light-house; then AB=60 miles, AC=20 miles, BC=70 miles. Again, from the bearings of the object, the ∠ASB=112° 30′, and ∠ASC=78° 45′; and since the sum of these angles is greater than two right angles, the point S is withing the triangle ABC. (Trig. Art. 97–94.)

Make the triangle ABC, having its sides respectively 60, 70, and 20 miles, as above, and at the point C, make the angle BCD= the supplement of the ∠ASB, and at the point B make the angle DBC= the supplement of the angle ASC; about the triangle DBC describe a circle, and join AD, which will cut the circle in S, and join BS, SC, and S will represent the position of Eddystone Light-house. For the angle BSD= the angle BCD= the supplement of BSA by construction, which it evidently is.

Now, in the △DBC there are given the ∠DCB=67° 30′, the ∠DBC=101° 15′; hence the ∠BDC=11° 15′, and BC=70.

∴ sin. 11° 15′ : sin. 101° 15′=BC : DC.

Log. cosec. 11° 15′ = 10·709764
Log. sin. 101° 15′ = 9·991574
Log. 70 = 1·845098
∴ DC=351·914 = 2·546436

Next, from the △ABC find the ∠ACB. (Trig. Art. 45.)

AC	= 20 ar. co. Log.	= 8·698970
BC	= 70 ,,	= 8·154902
AB	= 60	
	2)150	
S	= 75 Log. of	= 1·875061
S—AB	= 15 ,,	= 1·176091
		2)19·905024
Cos. $\frac{1}{2}$ACB=	26° 18′ 30″	= 9·952512
	2	
∴ ∠ACB	= 52° 37′	
∠BCD	= 67° 30′	
∴ ∠ACD	=120° 7′	

Again, in the △ACD there are given AC=20, DC=351·914, and the ∠ACD=120° 7′, to find the ∠s at A and D; now half the sum of the ∠s at A and D $=\frac{1}{2}$(180°—120° 7′)=29° 56′ 30″.

DC	=351·914	
AC	= 20	
Sum	=371·914 ar. co. Log.	= 7·429558
Difference,	=331·914 Log. of	= 2·521025
Log. tan.	29° 56′ 30″	= 9·760418
Log. tan.	27° 12′ 19″	= 9·711001
∴ ∠CAD	=57° 8′ 49″	
And ∠CDA=	2° 44′ 11″.	

But ∠BDC=11° 15′; ∴ ∠BDA=8° 30′ 49″, and the ∠BCS= the ∠BDA being in the same segment of a circle; and for the same reason, the ∠CDS= the ∠CBS; hence in the △SBC there are given the ∠s and the side BC to find BS, the distance of Eddystone from the Lizard, and SC, the distance of Eddystone from the Start; wherefore

Sin. S : sin. C=BC : BS, and
Sin. S : sin. B=BC : CS.

Log. cosec. 11° 15′	=10·709764 =	Ditto	=10·709764
Log. sin. 8° 30′ 49″	= 9·170392	Log. sin. 2° 44′ 11″	= 8·678890
Log. 70	= 1·845098	Log. 70	= 1·845098
∴ BS=53·12	= 1·725254	∴ CS=17·13	= 1·233752

Lastly, in the △ACS the ∠ACS=∠ACB—∠SCB= 52° 37′—8° 30′ 49″=44° 6′ 11″, and ∠ASC is given 78° 45′; ∴ the third ∠CAS=57° 8′ 49″, and AC=20 miles; *hence*

Sin. S : sin. C=AC : AS.

Log. cosec. 78° 45′ = 10·008426
Log. sin. 44° 6′ 11″ = 9·842579
Log. 20 = 1·301030
∴ AS=14·19 miles = 1·152035

(22.) Let x = those who voted for the first.
Then $x-22=$ those who voted for the second.
$x-73=$ those who voted for the third.
And $x-130=$ those who voted for the fourth.
∴ $4x-225=5219$, by the question.
$4x=5444$, by transposition.
∴ $x=1361$; hence $x-22=1339$, $x-73=1288$, and $x-130=1231$.

(23.) It is evident that a diameter drawn through the vertex will bisect the vertical angle, and that each of the = sides will be $=24\times$ cos. 15°; and (by Problem II. Rule II., Mens. Surf.) the area will be $\frac{1}{2}(24 \text{ cos. } 15°)^2$ sin. 30°; but sin. 30°=2 sin. 15° cos. 15° (Trig. 69); hence the area $=(24)^2(\text{cos. } 15°)^3$ sin. 15°; ∴ Log. area =2 Log. 24+3 Log. cos. 15°+ Log. sin. 15°—40.

2 Log. 24 = 2·760422
3 Log. cos. 15° = 29·954832
Log. sin. 15° = 9·412996
∴ area =134·354 = 2·128250

(24.) Let $s=$ the semiperimeter, and r the radius of the inscribed circle, it is evident that $sr=$ the area; therefore

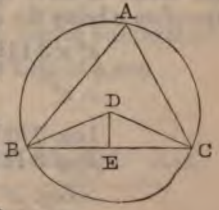
$$r=\frac{\text{area}}{s}$$

$$=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}. \quad \text{(Mens. p. 63.)}$$

$$=\sqrt{\frac{50\times30\times10}{90}}$$

$$=\sqrt{\frac{1500}{9}}=\tfrac{10}{3}\sqrt{15}=3\tfrac{1}{3}\sqrt{15}. \quad \text{Ans.}$$

Let ABC be the △, and D the centre of the circumscribing circle, also DE a perpendicular from the centre on BC, and having joined DB, DC,



It is plain that the ∠BDC is double of the angle BAC, or that the ∠BDE is = the ∠A; and ∴ R : $\frac{1}{\textit{sin. A}}$ $=BE : BD=\frac{\text{BE}}{\text{R sin. A}}=\frac{\text{BE}}{\text{sin. A}}$ to $R=1$.

But (Mens. Surf., p. 63) $\sin. A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$, and using the same notation, $BE = \frac{a}{2}$; therefore

$$BD = \frac{\frac{1}{2}a}{\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} =$$ the product of the three sides, divided by four times the area of the triangle. Substituting the values of a, b, c, and s, we have the radius

$$= \frac{40\times60\times80}{4\sqrt{90\times50\times30\times10}} = \frac{192000}{4\sqrt{1350000}}$$

$$= \frac{192000}{1200\sqrt{15}} = \frac{160}{\sqrt{15}} = \frac{160}{15}\sqrt{15}$$

$$= 10\tfrac{2}{3}\sqrt{15}. \quad \text{Ans.}$$

(25.) Let $x =$ the side of the triangle; then (page 63, cor., Mens. Surf.) the area $= \frac{x^2}{4}\sqrt{3}$; therefore by the question, $\frac{x^2}{4}\sqrt{3} = 200000$ links.

$$\therefore x^2 = \frac{800000}{\sqrt{3}}$$

$$\begin{array}{lr} \text{Log. } 800000 & = 5{\cdot}903090 \\ \tfrac{1}{2}\text{ Log. } 3 & = \ {\cdot}238560 \\ & 2\overline{)5{\cdot}664530} \\ \text{Log. } x = \text{Log. } 679{\cdot}619 & = 2{\cdot}832265 \\ \therefore x = \quad 679{\cdot}619. & \text{Ans.} \end{array}$$

(26.) It is evident that the quantity eaten will be the sector of a circle containing 60°, which is the sixth part of the whole circle; hence the area of the circle of which it is the sector is three times the area of the whole triangle; therefore x being the radius,

$$\begin{array}{lr} x^2\times3{\cdot}1416 & = \quad 7260 \\ \therefore x^2 & = \quad \frac{7260}{3{\cdot}1416} \\ \text{Log. } 7260 & = 3{\cdot}860937 \\ \text{Log. } 3{\cdot}1416 & = 0{\cdot}497151 \\ \therefore \text{Log. } x^2 & = 3{\cdot}363786 \\ x = 48{\cdot}072 & = 1{\cdot}681893 \end{array}$$

(27.) The other side $=\frac{2\times 60}{8}=15.$

Hypotenuse $=\sqrt{(15)^2+8^2}.$

$=\sqrt{289}=17.$ Ans.

(28.) Since the vertical angle is 3 times the angle at the base, 5 times the angle at the base is $=180^\circ$; $\therefore$ the angles at the base are each 36°, and the vertical angle is 108°, consequently half the vertical angle $=54^\circ$. Now if $x=$ the diameter of the circle, each of the equal sides will $=x$ cos. 54°, and as shown, (Ex. 23), the area of the triangle$=x^2(\cos. 54^\circ)^3 \sin. 54^\circ=2420$, by the question; $\therefore x^2=2420\ (\sec. 54^\circ)^3 \operatorname{cosec}. 54^\circ$.

Log. 2420 $=3\cdot383815$
3 (Log. sec. 54°—10) $=\cdot692343$
Log. cosec. 54°—10 $=\cdot092042$
2 Log. x $=2)4\cdot168200$
$\therefore x=121\cdot367$ $=2\cdot084100$

(29.) Surface$=4\times 24^2\times\cdot7854=1809\cdot5616$ sq. inches.
And 1809·5616 at 1½d. =L11, 6s. 2¼d. Ans.

(30.) $\cdot0087266\times 50=\cdot43633=\frac{1}{2}$ arc to R=1.

Hence if x the radius of the circle, we have by the question,

$x^2\times\cdot43633 = 1210$ yards.

$\therefore x^2 = \frac{1210}{\cdot43633}$

Log. 1210 $=3\cdot082785$
Log. ·43633 $=\bar{1}\cdot639815$
$2)3\cdot442970$
$\therefore x$ $=52\cdot6605=1\cdot721485$
$\bar{1}\cdot639815$ Log. of ½ arc to R=1.
$22\cdot9773=1\cdot361300$ Log. of ½ arc.
Semiperimeter 75·6378
2
151·2756 at 5s. =L.37, 16s. 4½d. Ans.

(31.)

$$\begin{array}{ll}
\text{Log. } \cdot 0174533 & = \bar{2}\cdot 241877 \\
\text{Log. } 70 & = 1\cdot 845098 \\
\text{Log. } 20 & = 1\cdot 301030 \\
\hline
\text{Log. diameter} & = 1\cdot 388005 \\
 & \qquad\qquad\; 2 \\
\hline
\text{Log. (dia.)}^2 & = 2\cdot 776010 \\
\text{Log. } \cdot 7854 & = \bar{1}\cdot 895091 \\
\hline
\therefore \text{ area } = 468\cdot 922 & = 2\cdot 671101
\end{array}$$

(32.) The circumference of a circle whose radius is 1 is 6·2832; again, the circumference of the second circle will be $(6\cdot 2832)^2$, which is the radius of the third; hence the area will be

$$\begin{array}{ll}
(6\cdot 2832)^4 \times 3\cdot 1416 & = 2^4 \times (3\cdot 1416)^5 \\
\text{Log. } 3\cdot 1416 & = 0\cdot 497151 \\
 & \qquad\qquad\; 5 \\
\hline
 & \quad 2\cdot 485755 \\
4 \text{ Log. } 2 & = 1\cdot 204120 \\
\hline
\therefore \text{ area } = 4896\cdot 38 & = 3\cdot 689875
\end{array}$$

(33.) By Exercise 7, page 37, we have

$4\cdot 5 : 5 = \cos. 36^\circ : \cos. \frac{1}{2}$ the difference of the angles at the base; hence

$$\begin{array}{ll}
\text{Ar. co. Log. } 4\cdot 5 & = 9\cdot 346787 \\
\text{Log. } 5 & = 0\cdot 698970 \\
\text{Log. cos. } 36^\circ & = 9\cdot 907958 \\
\hline
\text{Log. cos. } 25^\circ\ 59'\ 7'' & = 9\cdot 953715
\end{array}$$

$\therefore$ less $\angle = 10^\circ\ 0'\ 53''$.

And greater $\angle = 61^\circ\ 59'\ 7''$.

Again, $\sin. 25^\circ\ 59'\ 7'' : \sin. 36^\circ = 5$: sum of sides.

$$\begin{array}{ll}
\text{Log. cosec. } 25^\circ\ 59'\ 7'' & = 10\cdot 358387 \\
\text{Log. sin. } 36^\circ & = 9\cdot 769219 \\
\text{Log. } 5 & = 0\cdot 698970 \\
\hline
\therefore \text{ sum of sides } = 6\cdot 70774 & = 0\cdot 826576
\end{array}$$

Dif. of sides $= 4\cdot 5$.

$\therefore$ 2 the less $= 2\cdot 20774$.

The less $= 1\cdot 10387$.

And the greater $= 5\cdot 60387$.

To find the base, $\sin. 61^\circ\ 59'\ 7'' : \sin. 108^\circ = 5\cdot 60387$ *: the base.*

Log. cosec. 61° 59′ 7″ = 10·054124
Log. sin. 108° = 9·978206
Log. 5·60387 = 0·748488
∴ the base =6·03699 = 0·780818

Lastly, to find the area from the two sides and the contained angle, we have

Log. $\frac{1}{2}$(5·60387) = 0·447458
Log. 1·10387 = 0·042918
Log. sin. 108° = 9·978206
∴ area =2·94159 = 0·468582

(34.) Let $x=$ the diameter of the globe, then
The surface $=x^2\times 3{\cdot}1416$.
The solidity $=x^3\times\frac{3{\cdot}1416}{6}$.

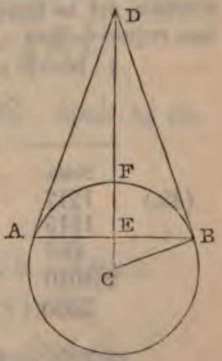
Hence $x^2\times 3{\cdot}1416=x^3\times\frac{3{\cdot}1416}{6}$, by the question.

$1=\frac{x}{6}$, by dividing by $x^2\times 3{\cdot}1416$.
$\therefore 6=x$, multiplying by 6.

(35.) Since the depth of the cistern is to be the same, it is plain that the area of the base must be three times as great, and that the new diameter will be the former multiplied into the square root of 3; hence the diameter $=26{\cdot}3\sqrt{3}=45{\cdot}553$ nearly.

(36.) $(20)^3\times{\cdot}5236\div(5^3\times{\cdot}5236)=64$. Ans.

(37.) It is plain, since the surface to be seen is $\frac{1}{3}$ of the whole surface, that FE$=\frac{1}{3}$ of the diameter, or $\frac{2}{3}$ of FC, (Mens. Solids, Prob. VIII.); and since DB touches the circle at B, the angle DBC is a $r^t\angle$; but the $\angle$BEC is a right angle, and the $\angle$ at C is common to each of the $\triangle$s DCB and BCE, therefore they are similar; hence EC : CB=CB : CD; but EC$=\frac{r}{3}$ and BC$=r$, $\therefore \frac{r}{3} : r=r :$ CD$=\frac{r^2}{\frac{1}{3}r}=3r$; $\therefore$ DF$=2r=$the diameter the height *sought.*



(38.) Surface of the ball$=3·1416\times(72)^2$.

$=16286·0544$ sq. inches.

16286·0544 at $3\frac{1}{2}$d.=L.237, 10s. 1d.+. Ans.

(39.) The upper third is obviously a cone similar to the original cone; therefore

$1:\frac{1}{3}=(18)^3:(h^t)^3=1944.$

$\therefore$ height $=\sqrt[3]{1944}=12·4805.$

Again, the two upper parts together will obviously also be a cone similar to the original cone; and hence

$1:\frac{2}{3}=(18)^3:(h^t)^3=3888.$

$\therefore$ height $=\sqrt[3]{3888}=15·7244.$

Hence the middle part$=15·7244-12·4805.$

$=3·2439.$

Also the height of the lower part $=18-15·7244.$

$=2·2756.$

(40.) Let $x=$ the length, then $(\frac{1}{30})^2\times·7854\times x=1728.$

$$\therefore x=\frac{1728\times900}{·7854}=1980138 \text{ inches.}$$

$=165012$ feet nearly.

(41.) Let $x=$ its diameter, then

$x^2\times·7854\times\frac{1}{192}=·5236.$

$\therefore x^2=\frac{2}{3}\times192=128.$

Therefore $x=\sqrt{128}=11·3137.$ Ans.

(42.) Since the altitudes of the columns are equal, their solidities are as their bases; let $x=$the diameter of the base required, then

$1:10=7^2:x^2=7^2\times10.$

$\therefore x=7\sqrt{10}=7\times3·162278$

$=22·135946$ inches.

	Sides.	Remainders.	Log.
(43.)	1717	303	= 2·481443
	1515	505	= 2·703291
	808	1212	= 3·083503
	2)4040		
	2020		= 3·305351
			2)11·573588
	612060 sq. links		= 5·786794

=6 acres 0 ro. 19 poles 8·954 yds.

NOTE—The above is a right-angled triangle, and hence its area ight have been found as follows:—

$$1515\times 404 = 612060 \text{ square links, as above.}$$

(44.) If $x=$ the side, then

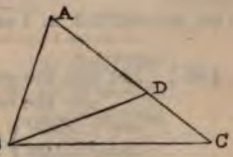
$$\text{the perp.} = \frac{x}{2}\sqrt{3} = 10.$$

$$\therefore x = \frac{20}{\sqrt{3}} = 6\tfrac{2}{3}\sqrt{3}.$$

Again, to find the diameter of the circle, we have

$$\text{(Geo., prop. 68) dia.} = \frac{x^2}{10} = \frac{1200}{90} = 13\tfrac{1}{3}. \quad \text{Ans.}$$

(45.) By (Art. 40, Note 2, rigonometry), $a : b-c = \sin. \frac{1}{2}$ B+C) : $\sin. \frac{1}{2}(B-C)$.



$$\begin{array}{ll}
\text{r. co. Log. } (a=160) & = 7{\cdot}795880 \\
\text{og. } \{(b-c)=8{\cdot}7365\} & = 0{\cdot}941338 \\
\text{og. sin. } \frac{1}{2}(B+C)=40^\circ & = \underline{9{\cdot}808067} \\
\text{. sin. } \frac{1}{2}(B-C)=2^\circ\, 0'\, 41'' & = 8{\cdot}545285
\end{array}$$

Hence $\angle B = 42^\circ\ 0'\ 41''$

nd $\angle C = 37^\circ\ 59'\ 19''$.

$$\begin{array}{llll}
\text{og. cosec. } 100^\circ & = 10{\cdot}006649 = & \text{Ditto} & = 10{\cdot}006649 \\
\text{og. sin. } 42^\circ\, 0'\, 41'' & = 9{\cdot}825607 & \text{Log. sin. } 37^\circ\, 59'\, 19'' & = 9{\cdot}789231 \\
\text{og. } 160 & = \underline{2{\cdot}204120} = & \text{Ditto} & = \underline{2{\cdot}204120} \\
\text{. AC} = 108{\cdot}7365 & = 2{\cdot}036376 & \therefore \text{AB} = 100 & = 2{\cdot}000000
\end{array}$$

(46.) Let $7x$, $5x$, and $3x$, represent the sides; then The area $= \frac{1}{2}\times 5x\times 3x\times \sin. 120^\circ$.

$$= 7\tfrac{1}{2}x^2 \sin. 60^\circ = \tfrac{15}{2}x^2\times\tfrac{1}{2}\sqrt{3}.$$

$$= x^2\times\tfrac{15}{4}\sqrt{3}.$$

Also 10 times the perimeter $=150x$; hence by the uestion,

$$x^2\times\tfrac{15}{4}\sqrt{3} = 150x.$$

$$\therefore x = \frac{40}{\sqrt{3}} = 13\tfrac{1}{3}\sqrt{3}.$$

Therefore the sides are $93\frac{1}{3}\sqrt{3}$, $66\frac{2}{3}\sqrt{3}$, and $40\sqrt{3}$.

And the area $= \frac{1}{2}\times 66\frac{2}{3}\sqrt{3}\times 40\sqrt{3}\times\frac{1}{2}\sqrt{3}$.

$$= \tfrac{1}{4}\times\frac{200}{3}\times 40\sqrt{3}\times 3 = 2000\sqrt{3}.$$

(47.) By Trigonometry, Art. 40, Note 2, we have, since half the sum of the angles at the base is $\frac{1}{2}(180°—80°)=50°$, $100:150=$ cos. $50°$: cos. $\frac{1}{2}$ difference of the angles at the base; hence

Ar. co. Log. 100 = 8·000000
Log. 150 = 2·176091
Log. cos. 50° = 9·808067
Log. cos. 15° 22′ 55″ = 9·984158
∴ greater ∠ 65° 22′ 55″
And less ∠ 34° 37′ 5″.

Wherefore the sides will be found as follows:

Log. cosec. 80	=10·006649	=Ditto	=10·006649
Log. sin. 65° 22′ 55″=	9·958614	Log. sin. 34° 37′ 5″=	9·754427
Log. 100	= 2·000000	Log. 100	= 2·000000
∴ Gr. side=92·313	= 1·965263	Less 57·6868	= 1·761076

(48.) Let $x=$ the greater side,
$y=$ the less,

$$\left.\begin{array}{l} x+y=300 \\ x-y=100 \end{array}\right\}\text{by the question.}$$

$\therefore x=200$,
And $y=100$.

There are now given two sides and the contained angle; and since the one side is double of the other, and the contained angle is 60°, the triangle is evidently right-angled; hence the remaining angles are 90° and 30°.

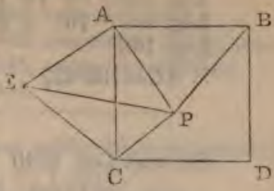
Also (Trig., Art. 40, Note 3), the third side is $=\sqrt{(200)^2+(100)^2-2\times200\times100\times\frac{1}{2}}=\sqrt{30000}=100\sqrt{3}$, and the area $=100\times100\times$ sin. 60°.

$=10000\frac{1}{2}\sqrt{3}$. (Trig. Art. 84.)
$=5000\sqrt{3}=8660\cdot254$. Ans.

(49.) $(20)^2\times\cdot7854\times\frac{50}{5}=400\times\cdot7854\times10$
$=3141\cdot6$ solid inches.
$=1\cdot81805$ solid feet.

(50.) $(10\times3-4\times2)\times4^2\times\cdot5236=$
$22\times16\times\cdot5236=184\cdot3072$ solid inches.

(51.) Let P represent the position of the oak, then CP=116, AP=156, and BP=166. Draw AE at $r^t\angle^s$ *to AP*, and make it =AP, and join EP; and from the *centre E*, with the radius BP, describe an arc, and from



e centre P, with the radius P, describe another arc, insecting the former in C, d join AC; AC is a side of e square.

Since EAP is a rt∠, and E is =AP, and ∴ EP= P × √2 = 156 × 1·4142136 = 220·617, &c.

From the △EPC, whose three sides are given, find the PEC. (Trig., Art. 45.)

$$\begin{array}{rll}
EP = & 220{\cdot}617. \text{ Ar. co. Log.} & = 7{\cdot}656360 \\
EC = & 166 \quad \text{ Ar. co. Log.} & = 7{\cdot}779892 \\
PC = & 116 & \\
& 2)\overline{502{\cdot}617} & \\
S = & 251{\cdot}3085 \text{ Log. of} & = 2{\cdot}400207 \\
S\text{—}PC = & 135{\cdot}3085 \quad ,, & = 2{\cdot}131325 \\
& & 2)\overline{19{\cdot}967784} \\
\text{Log. cos. } \tfrac{1}{2}PEC = & 15^\circ\ 30'\ 31'' & = 9{\cdot}983892 \\
& \qquad\qquad 2 & \\
\therefore \angle PEC & = 31^\circ\ 1'\ 2'' & \\
\text{But } \angle AEP & = 45^\circ\ 0\ .\ 0 & \\
\therefore \angle AEC & = 76^\circ\ 1'\ 2'' &
\end{array}$$

Wherefore half the sum of the ∠s EAC and ECA= 1° 59′ 29″.

Find now these angles from the △EAC, (Trig., Art. 40.)

$$\begin{array}{lll}
EC & = 166 & \\
EA & = 156 & \\
\text{Sum} & = 322 \text{ Ar. co. Log.} & = 7{\cdot}492144 \\
\text{Dif.} & = \ 10 \qquad \text{Log. of} & = 1{\cdot}000000 \\
\text{Log. tan.} & \ 51^\circ\ 59'\ 29'' & = 10{\cdot}107056 \\
\text{Log. tan.} & \ \ 2^\circ\ 16'\ 32'' & = 8{\cdot}599200 \\
\therefore \angle EAC & = 54^\circ\ 16'\ 1'' & \\
\text{And } \angle ECA = & 49^\circ\ 42'\ 57''. &
\end{array}$$

Lastly, from the △AEC find (Trig., Art. 39) the side AC, twice the logarithm of which diminished by the logarithm of 4840, the number of square yards in an acre, will be the logarithm of the area in acres.

Log. cosec. 54° 16′ 1″	=	10·090579
Log. sin. 76° 1′ 2″	=	9·986937
Log. 166	=	2·220108
∴ AC=198·437	=	2·297624
		2
		4·595248
Subtract Log. 4840	=	3·684845
∴ area =8·13585 acres	=	·910403

(52.) Content of the conical glass $=(2\frac{1}{4})^2\times\cdot7854\times\frac{2\frac{2}{5}}{3}$.

$=\frac{81}{16}\times\cdot7854\times\frac{4}{5}$.

Content of the frustum $=\{(3\frac{1}{2})^2+3\frac{1}{2}\times2+2^2\}$ $\times\cdot7854\times\frac{17}{12}$

$=\frac{93}{4}\times\cdot7854\times\frac{17}{12}$.

$\therefore\ \frac{\frac{93}{4}\times\cdot7854\times\frac{17}{12}}{\frac{81}{16}\times\cdot7854\times\frac{4}{5}}=\Big\}$ The times that the one contains the other.

$=\frac{93}{4}\times\frac{17}{12}\times\frac{16}{81}\times\frac{5}{4}$

$=\frac{31\times17\times5}{4\times81}=\frac{2635}{324}=8\frac{43}{324}$. Ans.

(53.) The number evidently

$=\frac{1728}{(\frac{1}{2})^3\times\cdot5236}$ the cubic inches in a foot. / the solidity of a ball.

$=\frac{1728\times8}{\cdot5236}=26401\cdot8$, &c. Ans.

(54.) Solidity of the cone $=1^2\times\cdot7854\times\frac{1}{3}=\frac{1}{3}\times\cdot7854$.

Solidity of the sphere $=6^3\times\cdot5236=6^3\times\frac{2}{3}\times\cdot7854$.

Hence the number $=\frac{216\times\frac{2}{3}\times\cdot7854}{\frac{1}{3}\times\cdot7854}$

$=216\times2=432$. Ans.

(55.) Let $x=$ the diameter of the bottom, then

The content $=\{(2\frac{1}{2})^2+2\frac{1}{2}x+x^2\}\times\cdot7854\times\frac{2}{3}$

$=(6\frac{1}{4}+2\frac{1}{2}x+x^2)\times\cdot5236=8$, by the question.

$\therefore\ x^2+2\frac{1}{2}x+6\frac{1}{4}=\frac{8}{\cdot5236}=15\cdot2788$.

$x^2+\frac{5}{2}x \quad = 9{\cdot}0288$, by transposing.
$x^2+\frac{5}{2}x+\frac{25}{16} = 10{\cdot}5913$, completing the square.
$x+\frac{5}{4} \quad = 3{\cdot}2544$, extracting the root.
$\therefore x \quad = 2{\cdot}0044$.

(56.) Each of the sides must evidently be increased in e ratio of $\sqrt[3]{1} : \sqrt[3]{4}$, which is 1 : 1·5874.

$$\left.\begin{array}{r}\therefore 1{\cdot}5874\times 4 = 6{\cdot}3496 \\ 1{\cdot}5874\times 3 = 4{\cdot}7622 \\ \text{And } 1{\cdot}5874\times 2\tfrac{1}{2} = 3{\cdot}9685\end{array}\right\}\text{The dimensions sought.}$$

(57.) To find the segments of the base made by the erpendicular from the vertex, we have, by (Geo., prop. 42, or.)

$$AD - DB = \frac{(25+20)(25-20)}{30}$$
$$= \frac{45\times 5}{30} = 7{\cdot}5.$$

ne half of which being added to half the base, gives the reater segment $=18{\cdot}75$, and subtracted from half the ase, gives the less segment 11·25.

To find the segments of the base made by the line bisectng the vertical angle, we have, by (Geo., prop. 63),

$(25+20) : 20 = 30 : 13\frac{1}{3}$, the less segment.
And $(25+20) : 25 = 30 : 16\frac{2}{3}$, the greater segment.
Lastly, by (Geo., prop. 43), we have

$$2CD^2 = (25)^2+(20)^2-2(15)^2$$
$$= 625+400-450 = 575.$$
$$\therefore CD = \sqrt{287{\cdot}5} = 17 \text{ nearly.}$$

(58.) Let x = the height.
$\therefore 2x$ = the breadth.
And $4x$ = the length.
Hence $8x^3$ = the solidity, and $6x\times 4x+2x\times x\times 2 = 28x^2$ = the surface; consequently
$8x^3 = 28x^2$, by the question.
$\therefore x = 3\frac{1}{2}$.
$2x = 7$.
And $4x = 14$.
$14\times 7\times 4 = 392$, number of people.

(59.) $5981{\cdot}16 \div \frac{370}{2} = 32{\cdot}3306$, radius of the inscribed *circle.*

And by Trigonometry, R : cot. 42° 43′ 47″ : : 32·3306 : the semiperimeter diminished by the side opposite to the given angle.

R = 10·000000
Cot. 42° 43′ 47″ = 10·034453
Log. 32·3306 = 1·509614
Log. 35 = 1·544067
∴ 35=185— the side sought.
Hence the side sought is =150.

SUPPLEMENT TO THE KEY.

(1.) *Lemma.* If $A+Bx+Cx^2+Dx^3+$ &c. $=a+bx$ $+cx^2+dx^3+$ &c., whatever be the value of x; then $A=a$, $=b$, $C=c$, &c.

By transposition, $(A-a)+(B-b)x+(C-c)x^2+$ $D-d)x^3+$ &c. $=0$.

Or, $(A-a)+x\{(B-b)+(C-c)x+(D-d)x^2+$ &c.$\}$ $=0$.

Now $A-a$ is either $=0$ or some constant, since both and a are constant; whereas $x\{(B-b)+(C-c)x+$ $D-d)x^2+$ &c.$\}$ is either $=0$, or varies with x, since x nters as a factor into it; and since neither $A-a=0$, nor $-a=$a constant, increased by a variable, can be equal to ; therefore $x\{(B-b)+(C-c)x+(D-d)x^2+$ &c.$\}=0$; ence $A-a+0=0$; $\therefore A=a$.

And it is evident that $x\{(B-b)+(C-c)x+(D-d)$ $^2+$ &c.$\}=0$, can only be true by $B-b=0$, or $B=b$; $C-c=0$, or $C=c$; $D-d=0$, or $D=d$; hence *when two xpressions containing x are always equal, the coefficients of he like powers of x are equal.*

(2.) *Proof of the Binomial Theorem.*

Let $(1+x)^n=1+Ax+Bx^2+Cx^3+$ &c., when n is a whole number.

Multiply both sides by $(1+x)$; then,

$$\begin{array}{l} (1+x)^{n+1}=1+Ax+Bx^2+Cx^3+\text{ \&c.} \\ \qquad\qquad\quad + \;\; x+Ax^2+Bx^3+\text{ \&c.} \\ \hline \qquad\qquad\quad =1+(A+1)x+\text{ \&c.,} \end{array}$$

where it is evident that the coefficient of x is increased by unity. If then in any one case the coefficient of x equals the index n, it is true in the next superior case; now it is true for $(1+x)^2$, (Alg. 29), and $\therefore$ for $(1+x)^3$, and therefore it is generally true.

Next, let the index be fractional.

And let $(1+x)^{\frac{m}{n}}=1+Ax+Bx^2+Cx^3+$ &c.

$(1+x)^m=(1+Ax+Bx^2+$ &c.$)^n$.

Or $1+mx+$&c. $=1+nAx+$ &c.

$$\therefore nA=m. \quad \text{Lemma (1.)}$$

$$\text{And } A=\frac{m}{n}.$$

$$\therefore (1+x)^{\frac{m}{n}}=1+\frac{m}{n}x+ \text{\&c.}$$

Again, let the index be negative.

$$(1+x)^{-n}=\frac{1}{(1+x)^n}=\frac{1}{1+nx+\text{\&c.}}=1-nx+\text{\&c.}$$

Hence the rule is generally true, whether n be integral, fractional, or negative.

Again, let

$(a+x)^n=a^n+na^{n-1}x+Pa^{n-2}x^2+Qa^{n-3}x^3+$ &c., where P and Q do not involve x.

$\therefore \{a+(x+z)\}^n=a^n+na^{n-1}(x+z)+Pa^{n-2}(x+z)^2+Qa^{n-3}(x+z)^3.$

$$\begin{aligned}&=a^n+na^{n-1}x+Pa^{n-2}x^2+Qa^{n-3}x^3+\text{\&c.}\\ &\quad+na^{n-1}z+2Pa^{n-2}xz+3Qa^{n-3}x^2z+\text{\&c.}\\ &\quad\quad+Pa^{n-2}z^2+3Qa^{n-3}xz^2+\text{\&c.}\end{aligned}$$

$\{(a+x)+z)\}^n=(a+x)^n+n(a+x)^{n-1}z+P(a+x)^{n-2}z^2+$&c.

$$=a^n+na^{n-1}x+Pa^{n-2}x^2+Qa^{n-3}x^3+\text{\&c.}$$
$$+na^{n-1}z+n(n-1)a^{n-2}xz+nP'a^{n-3}x^2z+nQ'a^{n-4}x^3z+\text{\&c.}$$

P′ and Q′ being what P and Q become when in them we put $(n-1)$ for n.

$\therefore$ equating the coefficients of the like powers of the combinations of x and z, we have

$$2P=n(n-1);\ \therefore P=\frac{n(n-1)}{1.2}.$$

$$\therefore P'=\frac{(n-1)(n-2)}{1.2}.$$

$$\text{Also } 3Q=nP'=\frac{n(n-1)(n-2)}{1.2}.$$

$$\text{And } Q=\frac{n(n-1)(n-2)}{1.2.3}.$$

$$\text{In the same way } R=\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.$$

$$\therefore (a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1.2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3+\text{\&c.}$$

(3.) *Exponential Theorem.*

It is required to expand a^x, or to find x in terms of a.

Let $a=1+b$, then $b=a-1$, and $a^x=(1+b)^x$; but by the binomial theorem,

$$(1+b)^x=1+bx+\frac{x(x-1)}{1.2}b^2+\frac{x(x-1)(x-2)}{1.2.3}b^3+\text{ \&c.}$$

Multiplying out the coefficients of the several terms, and then collecting the coefficients of x, we have

$(b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+$ &c.) for the coefficient of x, which call A, and assume the coefficients of x^2, x^3, &c., to be B, C, D, &c.; it is required to find the values of B, C, D, &c., in terms of A.

Let then

$$(1+b)^x=1+Ax+Bx^2+Cx^3+\text{ \&c.}$$

Then changing x into $2x$, we have

$$\begin{aligned}(1+b)^{2x}&=1+A(2x)+B(2x)^2+C(2x)^3+\text{ \&c.}\\&=1+2Ax+4Bx^2+8Cx^3+\text{ \&c.}\quad\text{(A.)}\end{aligned}$$

But this being the square of $(1+b)^x$, it must be

$=(1+Ax+Bx^2+Cx^3+$ &c.$)^2$; squaring, we have

$$\left.\begin{array}{rl}=1+Ax+Bx^2+Cx^3+\text{ \&c.}&\\ Ax+A^2x^2+ABx^3+\text{ \&c.}&\\ Bx^2+ABx^3+\text{ \&c.}&\\ +Cx^3+\text{ \&c.}&\end{array}\right\}\text{(B.)}$$

Equating now the coefficients of the like powers x in the equations (A) and (B), we have

$$2A=2A,\text{ or }A=A.$$

$$4B=2B+A^2.$$

$$\text{Or }2B=A^2;\ \therefore B=\frac{A^2}{2}.$$

$$\text{Also }8C=2C+2AB.$$

$$\text{Or }3C=AB=\frac{A^3}{2};\ \therefore C=\frac{A^3}{2.3}.$$

In the same manner, by expanding to another term, D would be found to $=\frac{A^4}{2.3.4}$, where the law of the formation of the coefficients is obvious, and $A=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+$ &c.

Hence $a^x=1+Ax+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\frac{A^4x^4}{1.2.3.4}+$ &c.

Let e be that value of a which will make $A=1$, then

$$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\text{\&c., and if } x=1,$$

$$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}+\text{\&c.}$$

To find e, we have

	1	1st term.
2)	1	2d ,,
3)	·50000000000	3d ,,
4)	·16666666666	4th ,,
5)	·04166666666	5th ,,
6)	·00833333333	6th ,,
7)	·00138888888	7th ,,
8)	·00019841269	8th ,,
9)	·00002480158	9th ,,
10)	·00000275573	10th ,,
11)	·00000027557	11th ,,
12)	·00000002505	12th ,,
13)	·00000000208	13th ,,
14)	·00000000016	14th ,,
	·00000000001	15th ,,

$\therefore e=2{\cdot}71828182847$, which is all correct except the last figure, which in adding was increased by 1 for every 9 in the last column, to compensate for what would have been to carry from the next column, had the expansion been carried farther.

(4.) *To find an expression for the logarithm of a number in terms of that number, and the base of the system.*

Let $a^x=N$, where N is any number, and x its logarithm to the base a; then

$a^{xy}=N^y$, and expanding both sides by the exponential theorem, we have

$$1+Axy+\frac{A^2x^2y^2}{1\cdot 2}+\text{\&c.}=1+A'y+\frac{A'^2y^2}{1.2}+\text{\&c.}$$

Equating the coefficients of the like powers of y, we have

$$Ax=A'.$$

$$\therefore x=\frac{A'}{A}.$$

But A' is the same function of N that A is of a;

$$\therefore x=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\frac{1}{4}(N-1)^4+\text{\&c.}}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\text{\&c.}}.$$

$$\text{Let } M=\frac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\text{\&c.}}.$$

Let also $N=1+n$, then $N-1=n$, and
x or $\text{Log.}\,(1+n)=M(n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.)$
Changing the sign of n in both sides, we have

$$\text{Log.}\,(1-n)=M(-n-\tfrac{1}{2}n^2-\tfrac{1}{3}n^3-\tfrac{1}{4}n^4-\&c.)$$

And by subtracting the second of these equations from the first, and observing that $\text{Log.}\,(1+n)-\text{Log.}\,(1-n)=$

$\text{Log.}\,\frac{1+n}{1-n}$, we have,

$$\text{Log.}\,\frac{1+n}{1-n}=2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\&c.)$$

(5.) Put $\frac{1+n}{1-n}=\frac{P+1}{P}$; $\therefore n=\frac{1}{2P+1}$, and therefore

$$\text{Log.}\,(P+1)-\text{Log.}\,P=2M\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\&c.\right)$$

from which we obtain this general expression for determining the logarithm of any number, $P+1$, when we know the logarithm of the preceding number P, viz.

$$\text{Log.}\,(P+1)=\text{Log.}\,P+2M\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\frac{1}{7(2P+1)^7}+\frac{1}{9(2P+1)^9}+\&c.\right)$$

If now we put $P=1, 2, 3, 4$, &c. successively, we get

$$\text{Log.}\,2=\text{Log.}\,1+2M\left(\frac{1}{3}+\frac{1}{3\,.\,3^3}+\frac{1}{5\,.\,3^5}+\frac{1}{7\,.\,3^7}+\&c.\right)$$

$$\text{Log.}\,3=\text{Log.}\,2+2M\left(\frac{1}{5}+\frac{1}{3\,.\,5^3}+\frac{1}{5\,.\,5^5}+\frac{1}{7\,.\,5^7}+\&c.\right)$$

$$\text{Log.}\,4=2\,\text{Log.}\,2.$$

$$\text{Log.}\,5=\text{Log.}\,4+2M\left(\frac{1}{9}+\frac{1}{3\,.\,9^3}+\frac{1}{5\,.\,9^5}+\frac{1}{7\,.\,9^7}+\&c.\right)$$

$$\text{Log.}\,6=\text{Log.}\,2+\text{Log.}\,3.$$

$$\text{Log.}\,7=\text{Log.}\,6+2M\left(\frac{1}{13}+\frac{1}{3\,.\,13^3}+\frac{1}{5\,.\,13^5}+\frac{1}{7\,.\,13^7}+\&c.\right)$$

(6.) Next, let $M=1$, then the logarithms will be calculated to the base $e=2{\cdot}718281828$, and are those called Naperian logarithms; and since the logarithm of 1 in every system is 0, we have

$$\text{Log. } 2 = 2\left(\frac{1}{3} + \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} + \frac{1}{7\,.\,3^7} + \&\text{c.}\right);$$

which can be calculated as follows:

$$\begin{array}{lrll}
3)2 & & & \\
9)\overline{\cdot 6666666666} & & = \cdot 6666666666 & \text{1st term.} \\
9)\overline{\cdot 0740740740} & \div\ 3 & = \cdot 0246913580 & \text{2d } ,, \\
9)\overline{\cdot 0082304526} & \div\ 5 & = \cdot 0016460905 & \text{3d } ,, \\
9)\overline{\cdot 0009144947} & \div\ 7 & = \cdot 0001306421 & \text{4th } ,, \\
9)\overline{\cdot 0001016105} & \div\ 9 & = \cdot 0000112900 & \text{5th } ,, \\
9)\overline{\cdot 0000112900} & \div 11 & = \cdot 0000010263 & \text{6th } ,, \\
9)\overline{\cdot 0000012544} & \div 13 & = \cdot 0000000965 & \text{7th } ,, \\
9)\overline{\cdot 0000001394} & \div 15 & = \cdot 0000000093 & \text{8th } ,, \\
\overline{\cdot 0000000155} & \div 17 & = \underline{\cdot 0000000009} & \\
\therefore \text{ Naperian Log. } 2 & & = \cdot 6931471805 &
\end{array}$$

(7.) Now three times the logarithm of 2 is the logarithm of 8; and the difference between the logarithms of 8 and 10 is equal to the difference of the logarithms of 4 and 5, since $\frac{8}{10}=\frac{4}{5}$; now the difference of the logarithms of 4 and 5 is

$$2\text{M}\left(\frac{1}{9} + \frac{1}{3\,.\,9^3} + \frac{1}{5\,.\,9^5} + \frac{1}{7\,.\,9^7} + \&\text{c.}\right)$$

which since M=1, can be calculated as follows:

$$\begin{array}{lrll}
9)2 & & & \\
81)\overline{\cdot 2222222222} & & = \cdot 2222222222\cdot & \text{1st term.} \\
81)\overline{\cdot 0027434842} & \div 3 & = \cdot 0009144947\cdot & \text{2d } ,, \\
81)\overline{\cdot 0000338701} & \div 5 & = \cdot 0000067740\cdot & \text{3d } ,, \\
81)\overline{\cdot 0000004181} & \div 7 & = \cdot 0000000597\cdot & \text{4th } ,, \\
\overline{\cdot 0000000051} & \div 9 & = \underline{\cdot 0000000005} & \\
 & & \ \cdot 2231435513 & \\
\cdot 6931471805 \times 3 & & = \underline{2\cdot 0794415417} & \\
\therefore \text{ Naperian Log. } 10 & & = 2\cdot 3025850930 &
\end{array}$$

(8.) In the common logarithms, since the base is 10, the logarithm of 10 is 1; therefore in order to find the *modulus* M of this system, we have the equation M × Naperian Log. 10 = common Log. 10.

$$\text{Or } 2\cdot 3025850930 \times \text{M} = 1.$$

$$\therefore \text{M} = \frac{1}{2\cdot 3025850930} = \cdot 4342944819.$$

And therefore 2M=·8685889638, which being used instead of 2 as the first dividend, and the calculations being performed exactly as above, will give common instead of Naperian logarithms; thus, to find the common logarithm of 2, we have

3)·8685889638 = 2M.
9)·2895296546 = ·2895296546 1st term.
9)·0321699616 ÷ 3 = ·0107233205 2d ,,
9)·0035744401 ÷ 5 = ·0007148880 3d ,,
9)·0003971600 ÷ 7 = ·0000567371 4th ,,
9)·0000441288 ÷ 9 = ·0000049032 5th ,,
9)·0000049032 ÷11 = ·0000004457 6th ,,
9)·0000005448 ÷13 = ·0000000419 7th ,,
9)·0000000605 ÷15 = ·0000000040 8th ,,
·0000000067 ÷17 = ·0000000004 9th ,,
∴ common Log. 2 = ·3010299957
= ·301030 to 6 figures.

Next, Log. 3 = Log. 2 + 2M $\left(\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \frac{1}{7.5^7} + \text{\&c.}\right)$

5)·8685889638 = 2M.
25)·1737177927 = ·1737177927
25)·0069487117 ÷ 3 = ·0023162372
25)·0002779484 ÷ 5 = ·0000555896
25)·0000111179 ÷ 7 = ·0000015882
25)·0000004447 ÷ 9 = ·0000000494
·0000000178 ÷11 = ·0000000016
·1760912590
Add Log. 2 = ·3010299957
∴ Log. 3 = ·4771212547
Also Log. 4=2 Log. 2 = ·6020599914
Log. 5= Log. 10— Log. 2
= 1—·3010299957= ·6989700043
Log. 6= Log. 2+ Log. 3 = ·7781512504

Log. 7 = Log. 6 + 2M $\left(\frac{1}{13} + \frac{1}{3.13^3} + \frac{1}{5.13^5} + \frac{1}{7.13^7} + \text{\&c.}\right)$, which may be calculated as shown above; and in the same manner we may proceed to any extent.

Those who wish to study this subject more at length, will find it amply discussed in Professor Young's *Elementary Essay on the* Computation of Logarithms.

Differential and Integral Calculus.

(9.) In the differential calculus the question to be solved is, At what rate does a given function of a variable quantity increase or decrease, when the variable on which it depends is increased or diminished by an infinitely small quantity.

(10.) Thus if $u=x^2$, u is called a function of x. Now if x receive a small increment dx, or become $x+dx$, we want to know what ratio the increase of u or du will bear to dx. In the above function, for u write $u+du$, and for x write $x+dx$; then

$$u+du=(x+dx)^2.$$

$$\therefore u+du=x^2+2xdx+dx^2, \text{ by squaring } (x+dx).$$

$$\text{But } u \quad =x^2.$$

$$\therefore \quad du=2xdx+dx^2, \text{ by subtraction.}$$

$$\text{Hence } \frac{du}{dx}=2x+dx, \text{ by dividing by } dx.$$

Now since dx may be as small as we please, the second side may be made as near to $2x$ as we please; it is therefore called the *limiting ratio*, or *differential coefficient*, and is the quantity sought.

$$\therefore \text{ if } u=x^2, \ \frac{du}{dx}=2x.$$

(11.) Next, let $u=ax^n$; then substituting as before, we have $u+du=ax^n+anx^{n-1}dx+\frac{an(n-1)}{1.2}x^{n-2}dx^2+$ &c.

$$\text{But } u \quad =ax^n.$$

$$\therefore du=anx^{n-1}dx+\frac{an(n-1)}{1.2}x^{n-2}dx^2+ \text{\&c.}$$

$$\text{Hence } \frac{du}{dx}=anx^{n-1}+a\frac{n(n-1)}{1.2}x^{n-2}dx+ \text{\&c.}$$

Which in the limit when $dx=0$, reduces to

$$\frac{du}{dx}=anx^{n-1}, \text{ or } du=anx^{n-1}dx.$$

But the differential coefficient is evidently the original expression multiplied by $\frac{n}{x}$: whence the following *Rule for differentiating any power of a variable.*

Multiply by the exponent of the variable, and diminish its exponent by unity, the result will be the differential coefficient required.

(12.) Cor. 1. Since the binomial theorem by which the above was expanded has been proved to be true, whether n be whole or fractional, positive or negative, the above rule must be true in all these cases likewise.

(13.) Cor. 2. If, instead of $\frac{du}{dx}$, we want to find du, the rule will be,

Multiply by the exponent, diminish the exponent by unity, and multiply by the differential of the variable.

(14.) Cor. 3. Since a, the constant multiplier of x^n, appears also as a constant multiplier in the differential coefficient, it follows that a times the differential coefficient of a quantity is the differential coefficient of a times that quantity.

But a variable quantity increased or diminished by a constant, has the same differential coefficient as the variable quantity itself; for

$u=x^2\pm c$, gives

$$u+du=x^2+2xdx+dx^2\pm c.$$

Subtracting $u=x^2\pm c$, we have

$$du=2xdx+dx^2$$

$=2xdx$ in the limit, which is the same as the differential of x^2.

(15.) *To find the differential of a product*, as $u=xy$, where x and y are both variables.

Let x become $x+dx$, and y become $y+dx$, then

$$u+du=(x+dx)(y+dy).$$

Or $u+du=xy+ydx+xdy+dxdy$.

But $u \quad =xy$.

$$\therefore du=ydx+xdy+dxdy.$$

Or $du=ydx+xdy$, in the limit.

Hence the Rule. Multiply each of the variables by the differential of the other, and the sum will be the differential required.

(16.) To differentiate a fraction whose numerator and denominator are both variable.

Let $u=\frac{x}{y}$, where x and y are both variable.

Then $uy=x$.

$\therefore ydu+udy=dx$, (15), $\therefore$ by transposition.

$$ydu=dx-udy.$$

$$\text{Or } du=\frac{dx-udy}{y}\quad \text{But } u=\frac{x}{y}.$$

$$\therefore du=\frac{dx-\frac{xdy}{y}}{y}=\frac{ydx-xdy}{y^2},$$

which is the denominator multiplied by the differential of the numerator, minus the numerator multiplied into the differential of the denominator, and the difference divided by the square of the denominator, which is the rule required.

(17.) From the above rules the following results can be obtained:—

$u=x^2$, $\therefore du=2xdx$.

$u=x^3$, $\therefore du=3x^2dx$.

$u=x^7$, $\therefore du=7x^6dx$.

$u=x^{-4}$, $\therefore du=-4x^{-5}dx=-\frac{4dx}{x^5}$.

$u=x^{-2}$, $\therefore du=-2x^{-3}dx=-\frac{2dx}{x^3}$.

$u=ax^5$, $\therefore du=5ax^4dx$.

$u=bx^{\frac{3}{2}}$, $\therefore du=\frac{3}{2}bx^{\frac{1}{2}}dx$.

$u=\sqrt{x}=x^{\frac{1}{2}}$, $\therefore du=\frac{1}{2}x^{-\frac{1}{2}}dx=\frac{dx}{2x^{\frac{1}{2}}}$.

$u=\sqrt[3]{ax}=a^{\frac{1}{3}}x^{\frac{1}{3}}$, $\therefore du=\frac{1}{3}a^{\frac{1}{3}}x^{-\frac{2}{3}}dx=\frac{a^{\frac{1}{3}}dx}{3x^{\frac{2}{3}}}$.

$u=\frac{v}{z}$, $\therefore du=\frac{zdv-vdz}{z^2}$.

$u=\frac{5x}{3x^2}=\frac{5}{3x}$, $\therefore du=\frac{0-5\times 3dx}{9x^2}=-\frac{5dx}{3x^2}$.

$u=x^{-\frac{1}{3}}+b$, $\therefore du=-\frac{1}{3}ax^{-\frac{4}{3}}dx=-\frac{adx}{3x^{\frac{4}{3}}}$.

(18.) To find the differential coefficient of an exponential function, such as $u=a^x$.

Let $u+du=a^{x+h}$, h being used for dx.

$$=a^x\times a^h.$$

$$=a^x\left(1+Ah+\frac{A^2h^2}{1.2}+\&c.\right)\quad \text{Art. 3.}$$

$$\therefore u+du=a^x+Aa^xh+\frac{A^2a^xh^2}{1.2}+\&c.$$

But $u=a^x$.

$$\therefore du=Aa^xh+\frac{A^2a^xh^2}{1.2}+\&c.$$

Hence $\frac{du}{dx}=Aa^x$ in the *limit,* where A is the Naperian garithm of the base a. (4.)

Wherefore $\frac{du}{dx}=a^x\,\text{Log.}\,a$; hence the following

Rule. The differential coefficient of an exponential ith a variable exponent, is obtained by multiplying the xponential by the Naperian logarithm of the base.

(19.) Cor. If the base be e, the base of the Naperian ogarithms, its logarithm will be 1; hence if $u=e^x$, $\frac{du}{dx}=e^x$, r $du=e^x dx$.

(20.) To find the differential of a logarithmic function, s $u=$ Log. x to the base e.

Let $u+du=\text{Log.}\,(x+h)=\text{Log.}\,x+\text{Log.}\left(1+\frac{h}{x}\right)$.

Or $u+du=\text{Log.}\,x+\frac{h}{x}-\frac{h^2}{2x^2}+$ &c. Art. 4.

$$\therefore du=\frac{h}{x}-\frac{h^2}{2x^2}=\frac{dx}{x}\text{ in the limit; therefore}$$

The differential of the logarithm of a function is equal o the differential of the function, divided by the function tself.

Note. If the logarithms used be not Naperian logarithms, the differential will be the differential of the function, divided by the unction and by the Naperian logarithm of the base.

(21.) To find the differential of the sine of an arc.

Let $x=$ the arc, which is the measure of the angle of which the sine is given; and let

$$u=\sin x,\text{ and }u+du=\sin(x+h).$$

$$\therefore du=\sin(x+h)-\sin x=2\sin\frac{dx}{2}\,.\,\cos\left(x+\frac{dx}{2}\right)\ (\text{Trig. }60.)$$

Hence $\frac{du}{dx}=\frac{\sin\frac{dx}{2}}{\frac{1}{2}dx}\,.\,\cos\left(x+\frac{dx}{2}\right)$.

But in the limit $\frac{\sin\frac{dx}{2}}{\frac{1}{2}dx}=1$, and $\cos\left(x+\frac{dx}{2}\right)=\cos x$;

$$\therefore \frac{du}{dx}=\cos x,\text{ or }du=\cos x\,dx.$$

Whence the differential of the sine of an arc is equal to cosine of the arc, into the differential of the arc.

(22.) To find the differential of the cosine of x.

Let $u = \cos x$, $\therefore u^2 = \cos^2 x$, and $1 - u^2 = 1 - \cos^2 x = \sin^2 x$.

But $1 - u^2 = \sin^2 x$ being differentiated, gives

$$-2udu = 2 \sin x \cos x.dx. \quad \text{(Arts. 10 and 21.)}$$

$$\therefore du = -\sin x.dx, \text{ by dividing by } -2u = -2 \cos x.$$

Whence the differential of the cosine of an arc is equal to minus the sine of the arc, into the differential of the arc.

(23.) To differentiate the tangent of an arc; or $u = \tan. x$.

$$\therefore du = d \tan x = d\frac{\sin x}{\cos x} = \frac{\cos^2 x dx + \sin^2 x dx}{\cos^2 x} \quad \text{Art. 16.}$$

$$= \frac{(\cos^2 x + \sin^2 x)dx}{\cos^2 x} \ \text{(Trig. 21.)} = \frac{dx}{\cos^2 x} = \sec^2 x dx.$$

(24.) To differentiate the cotangent of an arc, or $u = \cot x$.

$$\therefore du = d \cot x = d\frac{\cos x}{\sin x} = -\frac{(\sin^2 x + \cos^2 x)dx}{\sin^2 x}$$

$$= -\frac{dx}{\sin^2 x} = -\operatorname{cosec}^2 x dx.$$

(25.) To differentiate secant x, or $u = \sec x$.

$$\therefore du = d \sec x = d\frac{1}{\cos x} = \frac{\cos x \times 0 + \sin x dx}{\cos^2 x}. \quad \text{Art. 16.}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} dx = \sec x \tan x dx.$$

(26.) To differentiate cosecant x, or $u = \operatorname{cosec} x$.

$$\therefore du = d \operatorname{cosec} x = d\frac{1}{\sin x}. \quad \text{(Trig. 19.)}$$

$$= \frac{\sin x \times 0 - \cos x dx}{\sin^2 x}. \quad \text{Art. 16.}$$

$$= -\frac{\cos x.dx}{\sin^2 x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} dx$$

$$= -\operatorname{cosec} x \cot x dx.$$

(27.) To differentiate the arc in terms of its tangent, considered as the independent variable.

Let $\tan x = t$. Differentiating we have

$\operatorname{Sec}^2 x dx = dt$. (23.)

$$\therefore dx = \frac{dt}{\sec^2 x} = \frac{dt}{1 + \tan^2 x}. \quad \text{(Trig. 78.)}$$

Hence $dx = \dfrac{dt}{1+t^2} = dt - t^2 dt + t^4 dt - t^6 dt + t^8 dt -$ &c.

(28.) To find the sine and cosine of an arc, in terms the arc itself.

1. Since the sine of an arc becomes *minus* when the arc self becomes minus, and remains of the same magnitude, is can only take place by its being a function of the odd owers of the arc; and since when the arc becomes very mall, the sine becomes very nearly equal to the arc, we ay assume, that

Sin $x = x + ax^3 + bx^5 + cx^7 +$ &c., where the coefficients , b, c, &c., are independent of x.

2. Since the cosine of an arc retains the same sign, and of the same magnitude when the arc becomes minus, hich can only take place by it being a function of even owers of the arc; and since when the arc is 0 its cosine 1, (Trig. 72), we may assume that cos $x = 1 + Ax^2 +$ $x^4 + Cx^6 +$ &c., where the coefficients, A, B, C, &c., are dependent of x.

3. Since $1 = \sin^2 x + \cos^2 x$, (Trig. 21), we have

$$1 = (x + ax^3 + bx^5 + \text{\&c.})^2 + (1 + Bx^2 + Cx^4 + \text{\&c.})^2$$
$$= \begin{cases} \quad x^2 + 2ax^4 + (2b + a^2)x^6 + \text{\&c.} \\ 1 + 2Ax^2 + (2B + A)x^4 + (2C + 2AB)x^6 + \text{\&c.} \end{cases}$$

Therefore by Lemma (1), the sum of the coefficients of e like powers of x is equal to 0; hence

$$2A + 1 = 0, \therefore A = -\tfrac{1}{2}.$$

But by substituting $2x$ instead of x in the expression for e cosine, we obtain

$$\text{Cos } 2x = 1 - \tfrac{1}{2}(2x)^2 + B(2x)^4 + C(2x)^6 + \text{\&c.}$$
$$= 1 - 2x^2 + 16Bx^4 + 64Cx^6 + \text{\&c.}$$

But cos $2x = 2\cos^2 x - 1$. (Trig. 71.)

$$= 2(1 - \tfrac{1}{2}x^2 + Bx^4 + Cx^6 + \text{\&c.})^2 - 1.$$
$$= 2 - 2x^2 + (4B + \tfrac{1}{2})x^4 + (4C - 2B)x^6 + \text{\&c.} - 1.$$
$$= 1 - 2x^2 + (4B + \tfrac{1}{2})x^4 + (4C - 2B)x^6 + \text{\&c.}$$

Equating now these two values of cos $2x$, we obtain by emma (1),

$$16B = 4B + \tfrac{1}{2}, \therefore B = \frac{1}{1.2.3.4}.$$

$$64C = 4C - 2B.$$

$$\text{Or } 30C = -B = -\frac{1}{1.2.3.4}.$$

$$\therefore C = -\frac{1}{1.2.3.4.5.6}.$$

Whence $\cos x = 1 - \dfrac{x^2}{1.2} + \dfrac{x^4}{1.2.3.4} - \dfrac{x^6}{1.2.3.4.5.6}$ + &c., *where the* law of the coefficients is obvious.

Resuming now the expression for sin x, and differentiating it, we obtain

$$d(\sin x)=(1+3ax^2+5bx^4+7cx^6+\&c.)dx=\cos x dx.$$

And $\therefore =\left(1-\frac{1}{1.2}x^2+\frac{1}{1.2.3.4}x^4-\frac{1}{1.2.3.4.5.6}x^6\right.$ $\left.+\ \&c.\right)dx.$

Since these two expansions must be identical, we have

$$3a=-\frac{1}{1.2}, \therefore a=-\frac{1}{1.2.3}.$$

$$5b=\frac{1}{1.2.3.4}, \therefore b=\frac{1}{1.2.3.4.5}.$$

And $7c=-\frac{1}{1.2.3.4.5.6}, \therefore c=-\frac{1}{1.2.3.4.5.6.7}.$

Whence $\sin x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+\&c.$

And $\cos x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\&c.$

INTEGRAL CALCULUS.

(29.) The integral calculus treats of the processes by which a function may be found, such that, being differentiated, it shall produce a given differential. This function is called the integral of the given differential, and the process by which it is found is called integration.

(30.) When an integral is proposed, its differential may always be found by general rules. There is however no direct rule by which we may return from the differential to the integral, except the obvious one of retracing the steps by which the differential has been deduced from the integral. When these are distinctly indicated, there is no difficulty; but in general the steps of calculation by which the differential might have been deduced from an integral are obliterated or unknown; or the differential may not have been a direct result of differentiation, and then there will be no traces to show how we may pass from it to the integral.

(31.) The symbol used to indicate integration is $\int$, the *initial* letter of the word sum; thus $\int 2xdx$ means the

antity, which being differentiated, will produce $2xdx$; t this be Ax^n, where A and n are to be determined. ifferentiate this expression, and we have $nAx^{n-1}dx=2xdx$; $n-1=1$, or $n=2$, and $nA=2$, or $2A=2$, hence $A=1$, d $\therefore \int 2xdx=x^2\pm C$, where $\pm C$ is a constant added or btracted, and must be determined from the nature of the estion, because constants added or subtracted disappear differentiation. (Art. 14.)

(32.) From the above, and the inversion of the proesses, in (Art. 10 to 14), we derive the following rule:

To integrate a differential of one term, such as $ax^n dx$, we ust increase the exponent of the variable by a unit, and en divide by the new exponent and by dx.

Cor. *The integral of a series of terms, each of which is of e above form, will be the sum of the integrals of each term und separately by the above rule.*

(33.) Since (Art. 20) $d\,\text{Log.}(x+a)=\frac{dx}{x+a}$, we have onversely $\int\frac{dx}{x+a}=\text{Log.}\ (x+a)+C$, whence it appears, hat *when the numerator of a differential is the differential f the denominator, the integral is the Naperian logarithm f the denominator.*

(34.) Since (Art. 15), $d(uv)=vdu+udv$, u and v being unctions of x; whence $uv=\int vdu+\int udv$, or, by transpoition, $\int udv=uv-\int vdu$.

By means of this formula, one part of the integral of udv is obtained, and the finding of the other is made to depend on the integration of another differential. The process thus indicated is called *integration by parts*, and is often very useful.

(35.) Since (Art. 16) $d\frac{x}{y}=\frac{ydx-xdy}{y^2}$, we have conversely $\int\frac{ydx-xdy}{y^2}=\frac{x}{y}+C$.

In the same manner, by inverting the processes in (Arts. 21, 22, 23, 24, 25, 26, and 27), we obtain

$$\int\cos xdx=\sin x;\ \int a\cos axdx=\sin ax$$

$$\int\sin xdx=-\cos x;\ \int\sin 3xdx=-\tfrac{1}{3}\cos 3x$$

$$\int \sec^2 x dx = \tan x; \int \frac{2dx}{\cos^2 x} = 2 \tan x.$$

$$\int \text{cosec}^2 x dx = -\cot x; \int \frac{dx}{a \sin^2 x} = -\frac{1}{a} \cot x.$$

$$\int \sec x \tan x dx = \sec x; \int \sec 2x \tan 2x dx = \tfrac{1}{2} \sec 2x.$$

$$\int \text{cosec}\, x \cot x dx = -\text{cosec}\, x; \int \frac{\cos x dx}{a \sin^2 x} = -\frac{1}{a} \text{cosec}\, x.$$

$$\int dx = \int (dt - t^2 dt + t^4 dt - t^6 dt + t^8 dt - \&c.)$$

$$\therefore x = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \tfrac{1}{9}t^9 - \&c. \quad \text{(Art. 32, cor.)}$$

But x is the arc which measures the angle, and t is the tangent of the angle; therefore if we know the length of the tangent for any angle, we can find the length of the arc which measures it, and consequently the whole circumference.

Now it was proved in (Art. 85, Trigonometry), that $\tan 45° = 1$; therefore putting 1 for t in the second side of the above series, we get *arc of* 45° *to radius one* $= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.$

But as this result approximates very slowly, we must seek for some other value; now $\tan.^{-1}\frac{1}{2} + \tan.^{-1}\frac{1}{3} = 45°$; (where $\tan.^{-1}\frac{1}{2}$ means *the arc whose tangent is* $\frac{1}{2}$),

For if in (Trig. 73) we put $\tan a = \frac{1}{2}$, $\tan b = \frac{1}{3}$, we have

$$\tan (a+b) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = \tfrac{5}{5} = 1 = \tan 45°, \text{ (Art. 85).}$$

Hence by putting $t = \frac{1}{2}$, and then $t = \frac{1}{3}$ in the above series, and adding the results together, we may obtain the length of an arc of 45°, which is an eighth part of the whole circumference.

The most convenient series is obtained by taking four times the arc, whose tangent is $\frac{1}{5}$, which gives (Trig. 77.),

$4 \tan.^{-1}\frac{1}{5} = \dfrac{\frac{4}{5} - \frac{4}{125}}{1 - \frac{6}{25} + \frac{1}{625}} = \frac{120}{119}$; this tangent being greater than 1, we find by (Trig. 74), the tangent of the arc, which must be subtracted from this to leave exactly an arc of 45°; let this arc be x, then

$$\tan x = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \tfrac{1}{239}, \text{ (Trig. 74). whence we have}$$

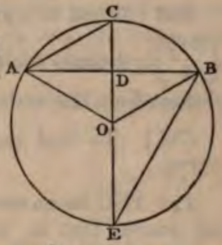
$$4 \tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239} = \tan^{-1} 1 = \tan 45;$$

And substituting these values in the above series, we

nd, that the arc. of $45° = 4(\frac{1}{5} - \frac{1}{3} \cdot \frac{1}{5^3} + \cdot \frac{1}{5} \cdot \frac{1}{5^5} - \frac{1}{7} \cdot \frac{1}{5^7} + \&c.)$
$-\left(\frac{1}{239} - \frac{1}{3} \cdot \frac{1}{239^3} + \frac{1}{5} \cdot \frac{1}{239^3} - \&c.\right)$
$= \cdot 785398163397$, &c., and therefore the semicircum-
erence to radius 1, or the whole circumference to the dia-
neter 1, is 3·141592653589, &c., which being taken to
he nearest figure to four decimal places, gives 3·1416, the
umber given as the circumference of the circle, when the
iameter is 1, in Prob. VII., page 68, Mensuration of Sur-
aces.

The multiplier given in Case II. is 1 divided by 3·1416,
nd the other numbers given in the rules are derived from
: by finding a fraction having a small numerator and de-
ominator, which shall be very nearly $= \frac{1}{3 \cdot 1416}$.

36.) PROOF OF PROBLEM VIII., page 69.



Let $x =$ the length of the arc ACB,
hen chord $AB = 2 \sin \frac{x}{2}$, and chord
$AC = 2 \sin \frac{x}{4}$; but (Art. 28),

$$2 \sin \frac{x}{2} = 2 \left\{ \frac{x}{2} - \frac{1}{1.2.3}\left(\frac{x}{2}\right)^3 + \frac{1}{.2.3.4.5}\left(\frac{x}{2}\right)^5 - \&c. \right\}$$

$$\therefore \text{chord } AB = x - \frac{x^3}{1.2.3.4} + \frac{x^5}{1.2.3.4.5.16} - \&c.$$

$$\text{And } 2 \sin \frac{x}{4} = 2 \left\{ \frac{x}{4} - \frac{1}{1.2.3}\left(\frac{x}{4}\right)^3 + \frac{1}{1.2.3.4.5}\left(\frac{x}{4}\right)^5 - \&c. \right\}$$

$$\therefore \text{chord } AC = \frac{x}{2} - \frac{x^3}{1.2.3.32} + \frac{x^5}{1.2.3.4.5.512} - \&c.$$

$$\text{Whence } 8AC = 4x - \frac{x^3}{1.2.3.4} + \frac{x^5}{1.2.3.4.5.64} - \&c.$$

$$\therefore 8AC - AB = 3x - \frac{3x^5}{1.2.3.4.5.64}.$$

$$\text{And } \frac{8AC - AB}{3} = x - \frac{x^5}{1.2.3.4.5.64}$$

$$= x - \frac{x^5}{7680} = x \text{ nearly}.$$

(37.) Proof of Problem VII., page 87.

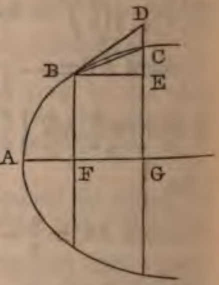
The prismoid is evidently the sum of 2 wedges, one of which has its base the greater end, and its edge the length of the less; and the other has its base the less end, and its edge the length of the greater, while the height of each is the length of the prismoid; hence if L= the length of the greater end, and l= the length of the less; B= the breadth of the greater end, and b= that of the less, and h= the common height, we have the solidity of the prismoid, by Problem VI.

$$\tfrac{1}{6}\mathrm{B}h(2\mathrm{L}+l)+\tfrac{1}{6}bh(2l+\mathrm{L})=$$
$$\frac{h}{6}(2\mathrm{LB}+l\mathrm{B}+2lb+\mathrm{L}b)=$$
$$\frac{h}{6}\left\{\mathrm{LB}+lb+4\left(\frac{\mathrm{L}+l}{2}\right)\left(\frac{\mathrm{B}+b}{2}\right)\right\}.$$

But LB and lb are the areas of the ends, and $4\left(\frac{\mathrm{L}+b}{2}\right)\left(\frac{\mathrm{B}+b}{2}\right)$ is 4 times the area of a section parallel to and equidistant from the ends; whence the rule is obvious.

(38.) To find the differential of an arc of a regular curve.

Let ABC be an arc of any regular curve, that is, any curve of which the line BF is the same function of AF and some constant, that CG is of AG and the same constant, and let AF=x, BF=y, then AG will $=x+dx$, and GC will $=y+dy$. Join BC, draw also BD touching the curve at the point B, and produce GC to meet it in D; draw also BE parallel to AG; then BE=dx, and EC=dy.



Now the nearer the point C is to B, the more nearly will the chord BC, and hence also the arc BC coincide with the tangent BD, which is their limiting direction; but the chord BC is the hypotenuse of a right-angled triangle, of which the sides are dx and dy; hence in the *limit* the differential of the arc is the same; hence calling the arc z, we have

$$dz=\sqrt{dx^2+dy^2}=\sqrt{1+\frac{dy^2}{dx^2}}\,dx.$$

If in this expression a value of $\frac{dy^2}{dx^2}$, found from the equation to the curve, be substituted, its integral will be the length of the curve sought.

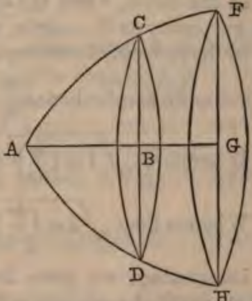
(39.) To find the differential of the area of a curve.

Let the notation be the same as in the last article; then the area of the quadrilateral BFGC will continually approach to the differential of the area, while the point G approaches to the point F; but the area of the trapezoid BFGC$=\frac{1}{2}$(BF$+$GC)$\times$FG, (Geo. prop. 33)$=\frac{1}{2}(y+y+dy)dx=ydx+\frac{1}{2}dydx$, which in the limit becomes ydx; therefore

$$d(\text{area})=ydx.$$

In the above expression the value of y must be found in terms of x, from the equation to the curve, and this value being substituted instead of it, and the expression integrated, the area of the curve will be obtained.

(40.) To find the differential of a surface of revolution.



Conceive the solid AFH to be generated by the revolution of the curve ACF about its axis AG; the differential of its surface will evidently be the differential of the arc, multiplied into the length of the path through which it passes, which path in the limit is the circumference of a circle whose radius is CB$=y$. Therefore if $\pi=3{\cdot}1416$, 2π will $=$ the circumference of a circle whose radius is 1, hence the circumference of the circle whose radius is $y=2\pi y$; but by (Art. 38), $dz=\sqrt{1+\frac{dy^2}{dx^2}}\,dx$; therefore calling the surface S, we have

$$ds=2\pi y\sqrt{1+\frac{dy^2}{dx^2}}\,dx,$$

in which the values of y and $\frac{dy^2}{dx^2}$ must be found from the equation to the curve, in terms of x, and substituted instead of them; after this is effected, the expression being integrated, will give the surface required.

(41.) To find the differential expression for a solid of revolution.

From the diagram, Art. 40, it is evident that the differential of the solidity, (which we shall represent by dv, V being the volume or solidity), will be greater than a cylinder, having for its base the circle generated by the revolution of $BC=y$, and for its height $BG=dx$, and that it will be less than a cylinder having for its base the circle generated by $GF=y+dy$, and for its height $BG=dx$; but this latter cylinder is $=\pi(y+dy)^2dx=\pi y^2dx+2\pi ydydx+\pi dy^2dx$, which as GF approaches to BC, becomes more nearly equal to πy^2dx, which is its limit; therefore

$$dv=\pi y^2dx,$$

where $\pi=3\cdot1416$, as in the last article. If now the value of y^2 in terms of x be found from the equation to the curve, and substituted instead of it, and the equation integrated, the solidity will be obtained.

(42.) Demonstration of the rule in Problem VIII., Mensuration of Solids.

In the circle, when the diameter is used as the axis, and its extremity as the origin, and the radius is represented by r, we have from Geometry, prop. 68,

$$y=\sqrt{2rx-x^2};\text{ hence }\frac{dy}{dx}=\frac{r-x}{\sqrt{2rx-x^2}}.$$

$$\text{Wherefore }\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{r}{\sqrt{2rx-x^2}}=\frac{r}{y}.$$

$$\text{Whence }2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=2\pi rdx=ds.\quad\text{(Art. 40.)}$$

Integrating, we have $2\pi rx=s$, the surface; and since when $x=0$, the surface is 0, it requires no correction. But $2\pi r=$ the circumference of the sphere, and x is the height of the part, which for the whole sphere is $2r$; whence the whole surface $=4\pi r^2=4$ times the area of a great circle.

Cor. The surface of a zone, contained between the heights $x=a$ and $x=b$, is evidently the difference of the surfaces $2\pi ra$ and $2\pi rb$, which if b be the greater, is equal to $2\pi rb-2\pi ra=2\pi r(b-a)=$ the circumference of a great circle multiplied into the height of the zone. Hence the rule is true, whether the surface required be a segment, a zone, or the whole sphere.

(43.) Demonstration of the rules in Problems IX. and X., Mensuration of Solids.

By (Art. 41) $dv = \pi y^2 dx$, and as shown in last article, y^2 in the circle $= 2rx - x^2$; substituting this value, we have

$$dv = \pi(2rx - x^2)dx$$
$$= 2\pi rxdx - \pi x^2 dx.$$

Integrating we have

$$V = \pi rx^2 - \tfrac{1}{3}\pi x^3$$
$$= \tfrac{1}{6}\pi(6r - 2x)x^2.$$

Which, since $\frac{1}{6}\pi = \frac{1}{6}(3{\cdot}1416) = {\cdot}5236$, $2r =$ the diameter of the circle or sphere, and $x =$ the height of the part, gives in its present form Rule II., Prob. X.

Again, $v = \frac{1}{6}\pi(6rx - 2x^2)x$, by taking one of the x's within the parenthesis; now the square of the radius of the base $= y^2 = 2rx - x^2$, hence

$$3y^2 + x^2 = 6rx - 3x^2 + x^2$$
$$= 6rx - 2x^2.$$
$$\therefore\ v = \tfrac{1}{6}\pi(3y^2 + x^2)x. \quad \text{Rule I., Prob. X.}$$

Next, in the expression $v = \frac{1}{6}\pi(6r - 2x)x^2$, put $x = 2r$, its value, for the whole sphere, and we have

$$v = \tfrac{1}{6}\pi(6r - 4r)4r^2 = \tfrac{1}{6}\pi \times 8r^2 = \tfrac{1}{6}\pi(2r)^3,$$

which is Rule II., Problem IX.

Lastly, Rule I., Problem IX., is found from the following consideration: the sphere may be conceived to be made up of an infinite number of small pyramids, the sum of whose bases will be the whole surface, and the common height the radius of the sphere; therefore their sum or the whole sphere will be the whole surface multiplied into $\frac{1}{3}r$, which is the rule.

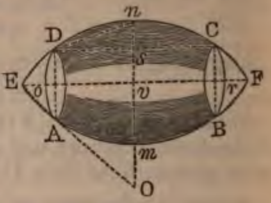
(44.) Demonstration of rule in Problem XI., Mensuration of Solids.

The difference of two segments of a sphere, whose heights are x and $x + h$, will evidently be the zone whose height is h; but by the last rule, the difference of these segments is

$$\frac{\pi}{6}\{6r(x+h) - 2(x+h)^2\}(x+h) - \frac{\pi}{6}(6rx - 2x^2)x$$
$$= \tfrac{1}{6}\pi(6rx^2 + 12rxh + 6rh^2 - 2x^3 - 6x^2h - 6xh^2 - 2h^3 - 6rx^2$$
$$+ 2x^3)$$
$$= \tfrac{1}{6}\pi(12rxh + 6rh^2 - 6x^2h - 6xh^2 - 2h^3)$$
$$= \tfrac{1}{6}\pi h\{3(2r\overline{x+h} - \overline{x+h}^2) + 3(2rx - x^2) + h^2\}.$$

But this latter expression is composed of three times the sum of the squares of the radii of the ends, together with the square of the height, multiplied by h, the height, and by $\frac{1}{6}\pi = \cdot 5236$, which is the rule; from which the rule given in Note I. is also obvious.

(45.) Demonstration of the rules in Problems XII. and XIII., Mensuration of Solids.



Let $\mathrm{O}v = c$, the central distance, ov or $\mathrm{D}s = x$, OE or $\mathrm{O}n = r$; then $\mathrm{D}o = \mathrm{O}s - \mathrm{O}v = \sqrt{r^2 - x^2} - c$; now (Art. 41) we have

$$\begin{aligned} dv &= 2\pi \mathrm{D}o^2 dx \\ &= 2\pi(\sqrt{r^2 - x^2} - c)^2 dx \\ &= 2\pi\left\{r^2 - c^2 - x^2 - 2c(\sqrt{r^2 - x^2} - c)\right\} dx \\ &= 2\pi\left\{(r^2 - c^2)dx - x^2 dx - 2c(\sqrt{r^2 - x^2} - c)dx\right\}. \end{aligned}$$

Therefore by integrating both sides, we have

$$\begin{aligned} v &= 2\pi\left\{(r^2 - c^2)x - \frac{x^3}{3} - c\,(\text{area } \mathrm{D}or\mathrm{C}n)\right\} \\ &= 2\pi\left\{(\mathrm{E}v^2 - \tfrac{1}{3}x^2)x - c \times \text{area } \mathrm{D}or\mathrm{C}n\right\}, \end{aligned}$$

which is the rule in Problem XIII.

Now if x in the above expression become $= \mathrm{E}v$, the area $\mathrm{D}or\mathrm{C}n$, will become the revolving segment $\mathrm{EF}n$, and the whole expression becomes the following:

$$v = 4\pi\left(\frac{\mathrm{E}v^3}{3} - \tfrac{1}{2}c \times \text{area of the segment } \mathrm{EF}n\right),$$

which is the rule in Problem XII.

Cor. If $c = o$, or the segment be a semicircle, then $\mathrm{E}v$ will be equal to r, and the spindle will become the whole sphere; but the expression then reduces to

$$v = 4\pi\frac{r^3}{3} = \tfrac{1}{6}\pi(2r^3), \text{ as proved in (43).}$$

Next, to prove the rule in Note I., Problem XII., using the same diagram, and making the centre the origin, we have, from the properties of the circle $\mathrm{D}o$, the revolving axis $= \sqrt{r^2 - x^2} - c$; also

$$d(\text{arc.}) = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx = \frac{r\,dx}{\sqrt{r^2 - x^2}}, \text{ whence}$$

$$ds=2\pi\left(\sqrt{r^2-x^2}-c\right)\frac{rdx}{\sqrt{r^2-x^2}}. \quad \text{(Art. 40.)}$$

$$=2\pi\left(rdx-c\times\frac{rdx}{\sqrt{r^2-x}}\right).$$

$\therefore S=2\pi\{rx-c\times(\text{arc. D}n)\}=$ the surface generated by the arc Dn, which when $x=0$, becomes 0, and therefore requires no correction; whence the whole surface generated by the revolving arc DnC will be double of the above; but the double of x is the chord DC, and the double of the arc Dn is the arc DnC, therefore the surface of the middle zone $=2\pi\{r\times\text{DC}-c\times(\text{arc D}n\text{C})\}$.

And for the whole surface of the spindle the expression will become $=2\pi\{r\times\text{EF}-c\times(\text{arc E}n\text{F})\}$, which is the rule.

Cor. If the central distance c vanish in the above expression, EF will become $=2r$, and the spindle will become the whole sphere; but in this case it reduces to $4\pi r^2$, the same as was proved in (42).

(46.) To prove the rules in Problem XIV., Mensuration of Solids.

Conceive the ring to be stretched out into a cylinder, its length will be the circumference of a circle, whose diameter is the inner diameter of the ring, increased by the thickness of the ring, and its solidity will evidently be the solidity of the cylinder, and its surface the surface of the cylinder.

Let now $a=$ the inner diameter, $b=$ the thickness of the ring, $\pi=3{\cdot}1416$, then $\frac{1}{4}\pi={\cdot}7854$, and the length of the cylinder will be $\pi(a+b)$, while the area of its end will be $\frac{1}{4}\pi b^2$; consequently the solidity will be $\frac{1}{4}\pi b^2\times\pi(a+b)=\frac{1}{4}\pi^2(a+b)b^2=2{\cdot}4674(a+b)b^2$, which is the rule.

Next, the surface of the ring will evidently be the circumference of the cylinder $=\pi b$, multiplied into the length of the cylinder, which was shown to be $\pi(a+b)$; hence the surface of the ring will be $=\pi^2(a+b)b=9{\cdot}8696(a+b)b$, which is the rule given for finding the surface.

(47.) Demonstration of the rule in Problem V., Specific *Gravities.*

Let $m =$ the weight of the whole compound mass, $x =$ the weight of the heavier ingredient, then $m-x$ will be the weight of the lighter; also let $s =$ the specific gravity of the heavier, s' that of the lighter, and s'' that of the compound mass; then $\frac{x}{s}$ will $=$ the solid content of the heavier body in feet, $\frac{m-x}{s'}$ that of the lighter, and $\frac{m}{s''}$ will be the solid content of the mass; but it is evident that the two former must be together equal to the latter; hence

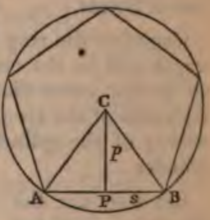
$$\frac{x}{s}+\frac{m-x}{s'}=\frac{m}{s''}.$$

Or $s's''x+ss''m-ss''x=ss'm.$

$$s''(s'-s)x=ms(s'-s'').$$

$$\left.\begin{aligned} \therefore x&=\frac{ms(s'-s'')}{s''(s'-s)} \\ \text{Similarly } m-x&=\frac{ms'(s''-s)}{s''(s'-s)} \end{aligned}\right\} \text{which is the rule.}$$

(48.) Demonstration of the rules for finding the areas of polygons, Mensuration of Surfaces, page 67.



Let $p =$ the perpendicular from the centre upon a side, $s =$ the length of a side, and $n =$ the number of sides.

The angle ACP is evidently the half of ACB, and therefore a $2n$th part of 360°; hence

$$\angle \text{ACP}=\frac{360^\circ}{2n}=\frac{180}{n}.$$

$$\therefore p=\tfrac{1}{2}s \text{ cot. } \frac{180}{n}, \text{ (Trig. 9), and}$$

$$\triangle\text{ACB}=p\times\frac{s}{2}=\frac{s^2}{4}\text{ cot. }\frac{180}{n}, \text{ therefore}$$

$$n\triangle\text{ACB}= \text{the polygon } =n\frac{s^2}{4}\text{ cot. }\frac{180}{n}=n\left(\frac{s}{2}\right)^2\text{cot. }\frac{180}{n}.$$

Whence Log. (area) $=2$ Log. $\frac{s}{2}+$ Log. $n+$ Log. cot. $\frac{180}{n}$ Rule II.

The areas or tabular multipliers given in the table are *the* areas of the polygons calculated by the above formula *to* $s=1$; then since it was proved in Geometry, prop. 71, that polygons are to one another as the squares of their corresponding sides, the areas of all similar polygons can be found from these by multiplying the tabular area by the square of the length of the side; for $1^2 : s^2 ::$ Tab. area : s^2 Tab. area, which is Rule III.

THE END.

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ERRATA IN PART II.

Page 14, line 11 from the bottom, *for* 11″, *read* 13.″
27, „ 2, *for* 59″ *read* 9″, and *for* 159·022, *read* 195·022.
28, „ 10 from bottom, *for* ½(80, &c., *read* ½(180°, &c.
29, „ 18 from bottom, *for* 998·161. *read* 998·164.
33, „ 15, *for* 24″, *read* 34″.
45, „ 2 of Table, *for* 73° *read* 75°, and *for* 3907·2, *read* 3946; also line 3, *for* 4257·9, *read* 4357·9.
48, „ 24, *for* 289·804, *read* 389·804.
70, „ 11, *for* $\frac{1}{2}\,\frac{AD^2+CD^2}{2CD}$, read $\frac{AD^2+CD^2}{2CD}$.
75, „ 5, *for* $\frac{156\times36+4\times(26)^2}{208}$, *read* $\frac{156\times36-4\times(26)^2}{208}$.
80, „ 16, *for* 16·89849, *read* 16·89949.
161, Ans. to Ex. 1, *for* 7·272 poles, *read* 6·272 poles.
Tables, page 41 line 1, *for* 54° *read* 24°.

ERRATA IN KEY.

Page 22, last solution, *for* 73° 13′ *read* 75° 13′, and correct the consequences.